Microeconomic Theory I: Lecture 5

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Envelope theorem

Consider the parametric maximization problem of choosing an optimal $x \in \mathbb{R}^L$:

$$\max_x f(x; q)$$

subject to

$$g_m(x; q) = 0 \text{ for } m \in \{1, ..., M\},$$

where $q = (q_1, ..., q_S) \in \mathbb{R}^S$.
Let $x(q)$ denote the optimal choice of x and $z(q)$ the value function

$$z(q)) := f(x(q); q).$$

The Lagrangean for this problem is:

$$L(x, \lambda; \bar{q}) = f(x; \bar{q}) - \sum_{m=1}^{M} \lambda_m g_m(x; \bar{q}).$$

Envelope theorem

Theorem (Envelope Theorem)

Let z be differentiable at $\bar{q}$. Then

$$\frac{\partial z}{\partial q_s} = \frac{\partial f(x(\bar{q}), \bar{q})}{\partial q_s} + \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}), \bar{q})}{\partial q_s}.$$

Proof Since $z(q)) = f(x(q); q)$ for all q, we have by chain rule

$$\frac{\partial z}{\partial q_s} = \frac{\partial f(x(\bar{q}), \bar{q})}{\partial q_s} + \sum_{l=1}^{L} \left(\frac{\partial f(x(\bar{q}), \bar{q})}{\partial x_l} \frac{\partial x_l(\bar{q}),}{\partial q_s} \right).$$

First order condition for the maximization problem gives:

$$\frac{\partial f(x(\bar{q}), \bar{q})}{\partial x_l} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{q}), \bar{q})}{\partial x_l}.$$

Combining and changing the order of summation, we have:

$$\frac{\partial z}{\partial q_s} = \frac{\partial f(x(\bar{q}), \bar{q})}{\partial q_s} + \sum_{m=1}^{M} \lambda_m \sum_{l=1}^{L} \left(\frac{\partial g_m(x(\bar{q}), \bar{q})}{\partial x_l} \frac{\partial x_l(\bar{q}),}{\partial q_s} \right).$$

But since $g_m(x(q), q) = 0$ for all q, we have for all m:

$$\sum_{l=1}^{L} \left(\frac{\partial g_m(x(\bar{q}), \bar{q})}{\partial x_l} \frac{\partial x_l(\bar{q}),}{\partial q_s} \right) = \frac{\partial g_m(\bar{q}), \bar{q})}{\partial q_s}.$$

Combining these equalities completes the proof. **QED**

Digression on Envelope Theorem

Consider again the parametric maximization problem

$$z(q) = sup_{x \in X(q)} f(x; q).$$

Assume i) $f(x; q)$ is continuous in x and differentiable in q for all $x \in X$ with a uniformly bounded derivative. ii) $X(q)$ is a non-empty valued and compact valued continuous correspondence in q.

Let $X^*(q)$ denote the (non-empty) set of maximizers and let $x^*(q) \in X^*(q)$ be an arbitrary selection of maximizers. Milgrom-Segal envelope theorem assumes that $q \in [\underline{q}, \bar{q}] \subset \mathbb{R}$ and proves that $z(q)$ is absolutely continuous and differentiable almost everywhere and that for all q,

$$d^- z(q) = \lim_{q \to q^-} \frac{\partial f(x(q), q)}{\partial q} \leq \lim_{q \to q^+} \frac{\partial f(x(q), q)}{\partial q} = d^+ z(q),$$

where $d^- z(q)$is the derivative of z at q from the left and $d^+ z(q)$is the derivative of z at q from the right.

Digression on Envelope Theorem

Hence whenever $z'(q)$ exists, we have

$$z'(q) = \frac{\partial f(x(q), q)}{\partial q} \text{ and } z(q) = z(\underline{q}) + \int_{\underline{q}}^{q} z(s)ds.$$

This will be used repeatedly in Microeconomic Theory IV

Notice that z fails to be differentiable only if there are multiple maximizers.

The envelope theorem is valid for all compact X hence also for finite X. We do not need differentiability in x for the result.

If each $f(x, q)$ is convex in q, then $z(q)$ is also convex. If each is concave, then the minimum value function $\underline{z}(q) := \min_{x \in X(q)} f(x, q)$ is also concave. These will be used very soon...

Indirect utility function $v(p, m)$

We let $v(p, w)$ denote the value function of the UMP. In other words,

$$v(p, w) := \max_{x \in B(p,w)} u(x) = u(x(p, w)).$$

Since u is a continuous function and $B(p, w)$ is a continuous correspondence, we can apply Berge's maximum theorem to conclude that v is a continuous function of (p, w).
Envelope theorem tells us that at the points where $v(p, w)$ is differentiable, we have:

$$\frac{\partial v(p, w)}{\partial p_i} = -\lambda x_i \text{ and } \frac{\partial v(p, w)}{\partial w} = \lambda,$$

where λ is the Lagrange multiplier of the UMP.

Notice that if both u and $x(p, w)$ are differentiable, then v is also differentiable.

Since $x(p, w)$ is the endogenous solution to the UMP, it would be better to have a condition for differentiability in terms of the primitives of the problem, i.e. in terms of u directly.

A sufficient condition for this is that u be strictly convex and twice continuously differentiable (as a function of x).

What other properties of $v\left(p, w\right)$ are implied by the utility maximization problem?

Conversely, what properties of $v\left(p, w\right)$ guarantee the existence of a utility function $u\left(x\right)$ that results in value function $v(p, w)$?

Properties of $v(p, w)$

Proposition (Properties of v)

Suppose that the utility function in a UMP is locally non-satiated. Then the associated indirect utility function satisfies the following:

1. *Homogenous of degree 0 in* (p, w).
2. *Strictly increasing in* w, *non-decreasing in* p_l *for all* l.
3. *Continuous in* (p, w).
4. *Quasiconvex in* (p, w).

A more advanced result by Diewert (1974) states that for any $v(p, w)$ satisfying these properties, there exists a locally non-satiated and quasiconcave $u(x)$ such that $v(p, w)$ is the value function of the UMP.

Recovering $u(x)$ from $v(p, w)$

For locally non-satiated preferences, $p \cdot x = w$ at the optimal solution of the UMP and so we could write $v(p, w) = v(p, p \cdot x)$.

Proposition

Assume that u is quasiconcave and has strictly positive partial derivatives and let $v(p, 1) = max_{x \in B(p, p \cdot x)}$. *Then for all* $x >> 0$, $v(p, p \cdot x)$ *achieves a has a minimum for some* $p >> 0$ *and*

$$u(x) = \min_{p \in \mathbb{R}^L_{++}} v(p, p \cdot x).$$

Recovering $u(x)$ from $v(p, w)$

Proof.

By definition, we have

$$u(x) \leq v(p, p \cdot x)$$

for all p. Therefore we only need to show that $u(x) = v(p, p \cdot x)$ for some $p >> 0$.

Take any $x^0 >> 0$ and let $p^0 = \nabla u(x^0) >> 0$. Set $\lambda^0 = 1$ and $p^0 \cdot x^0 = w^0$. Then (x^0, λ^0) satisfies the first order conditions for the UMP at (p^0, w^0). Since $u(x)$ is quasiconcave, the first order conditions are also sufficient. But then $u(x^0) = v(p^0, p^0 \cdot x^0)$. ∎

An Application: Inverse demands

Since the indirect utility function is homogenous of degree zero in p we normalize $p \cdot x = 1$ for now and we write $v(p, 1)$. With this normalization, we have uniqueness of minimizers in the previous minimization problem and we can use e.g. envelope theorem in our analysis.

In intermediate microeconomics, it is often convenient to use *inverse demand functions*. How can we get the inverse demand for good l, denoted by $p_l(x)$ for our multi-dimensional maximization problem? Denote by $p(x)$ the solution to $min_{\{p:p \cdot x=1\}} v(p, 1)$.

Theorem (System of inverse demands)

Let $u(x)$ have strictly positive partial derivatives. The inverse demand for good l at $w = 1$ is given by:

$$p_i(x) = \frac{\frac{\partial u(x)}{\partial x_i}}{\sum_{l=1}^{L} x_l \frac{\partial u(x)}{\partial x_l}}.$$

Inverse demands

Proof.

By the previous theorem, we have

$$u(x) = v(p(x), 1) = min_{p \in \mathbb{R}^L_{++}} v(p, 1) \text{ s.t. } p \cdot x = 1.$$

Consider the associated Lagrangian:

$$L(p, \lambda) = v(p, 1) - \lambda(1 - p \cdot x).$$

By envelope theorem, we have

$$\frac{\partial u(x)}{\partial x_l} = \lambda p_l(x) \text{ for } l = 1, ..., L.$$

Multiplying by x_l on both sides and summing over l gives

$$\sum_{l=1}^{L} x_l \frac{\partial u(x)}{\partial x_l} = \lambda p(x) \cdot x = \lambda.$$

Combining these two lines gives the result. ∎

Roy's identity

Proposition (Roy's Identity)

Given an indirect utility function $v(p, w)$, the Walrasian demands $x(p, w)$ can be recovered from

$$x_l(p, w) = -\frac{\frac{\partial v(p,1)}{\partial p_l}}{\frac{\partial v(p,1)}{\partial w}}.$$

This follows from the fact that $\frac{\partial v(p,w)}{\partial p_l} = \nabla u(x) \cdot D_{p_l} x(p, w) = \lambda p_l \cdot D_{p_l} x(p, w) = -\lambda x_l(p, w)$ where the last equality comes from Cournot aggregation.

Expenditure Minimization

In this lecture,we go over the main properties of the optimal solutions and the associated value function of the following expenditure minimization problem (EMP):

$$\min_{x \in \mathbb{R}^L_+} \; p \cdot x$$
$$\text{s.t. } \; u(x) \geq u.$$

Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}^L_{++}$. (we assume this throughout).

In Lecture 6, we provide more applications of the methods.

Expenditure Minimization

The solution to this problem $h(p, u)$ is called the Hicksian or compensated demand function.
The value function of EMP by $e(p, u)$ is called the expenditure function.
The following 'duality' is key to the analysis that follows:

Proposition

Fix a price vector $p \in \mathbb{R}^L_{++}$ *and a locally non-satiated utility function* $u(x)$.
i) If $x^* = x(p, w)$, *then* $x^* = h(p, u(x^*)) = h(p, v(p, w))$.
ii) If $x^* = h(p, u)$, *then* $x^* = x(p, p \cdot x^*) = x(p, e(p, u))$.

Proof.

By picture. ∎

Expenditure Minimization

Summarizing:

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, u) = x(p, e(p, u)).$$

Obviously then also:

$$w = e(p, v(p, w)) \text{ and } u = v(p, e(p, u)).$$

Proposition

(Properties of $e(p, u)$)
i) $e(p, u)$ is homogenous of degree 1 in p.
ii) Strictly increasing in u and non-decreasing in p_l for all l.
iii) Concave in p.
iv) Continuous in p, u.

Expenditure Minimization

Given the expenditure function, Hicksian demands can be obtained as follows:

Proposition

Suppose $u(x)$ *is continuous and strictly quasiconvex. Then, for all* (p, w)

$$h(p, u) = \nabla_p e(p, u).$$

Properties of Hicksian demand

- Adding up: $p \cdot h(p, u) = w$
- Homogeneity of degree 0 in prices: $h(\alpha p, u) = h(p, u)$ for all p, u, and scalars $\alpha > 0$
- Convexity: if $\succeq$ is convex, then $h(p, u)$ is a convex set; if $\succeq$ is strictly convex, then $h(p, u)$ is a function
- Matrix $D_p h(p, u)$: negative semidefinite, symmetric, and satisfies $D_p h(p, u) p = 0$

Observable implications from EMP

$D_p h(p,u)$ can be computed from $x(p,w)$ which is observable and thereby potentially testable

How to express $D_p h(p,u)$ in terms of $x(p,w)$?

Recall :

$$h(p,u) = x(p, e(p,u)).$$

Therefore (Slutsky Equation):

$$\begin{aligned} & D_p h(p,u) \\ = \; & D_p x(p, e(p,u)) + D_w x(p, e(p,u)) D_p e(p,u) \\ = \; & D_p x(p, e(p,u)) + D_w x(p, e(p,u)) h(p,u)^T \\ = \; & D_p x(p,w) + D_w x(p,w) x(p,w)^T, \end{aligned}$$

where we have set $w = e(p,u) = e(p, v(p,w))$.