Microeconomic Theory I: Lecture 7-8

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Producer Theory

Objectives in this class

- as before, we start with a single agent decisions facing given prices
- production set describes technology, not resources
- comparative statics involve only substitution effects, no wealth effects
- exogenous variables: prices
- endogenous variables: outputs and input demands

Primitives

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The model

1. Commodity Space $X = \mathbb{R}^L$
   - In contrast to Consumer theory, we have negative numbers possible.
   - for inputs $y_i < 0$,
   - for outputs $y_i > 0$.
2. Production Set
   - Summary of the technologically feasibe.
   - A subset $Y \subset \mathbb{R}^L$
3. Behavioral Assumption: profit maximization in Y
   - With linear prices, profit is $p \cdot y$

Assumptions on Y

(These are selectively assumed)

1. Y is non-empty and closed.
2. $Y \cap \mathbb{R}^L_+ = \{0\}$. (No free lunch, inaction possible).
3. $Y - \mathbb{R}^L_+ \subset Y$. (Free disposal).
4. $y \in Y \backslash \{0\} \implies -y \notin Y$. (Irreversibility).
5. Returns to scale:
   - 5.1 (DRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [0, 1]$.
   - 5.2 (IRS) $y \in Y \implies \alpha y \in Y$ for all $\alpha \in [1, \infty)$.
   - 5.3 (CRS) $y \in Y \implies \alpha y \in Y$ for all α.
6. $Y + Y \subset Y$ (Free entry).
7. Y is convex.

Alternative description of Y

Define a transformation function F

$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.

The boundary $\partial Y = \{y \in \mathbb{R}^L : F(y) = 0\}$ of the production set Y is called the transformation frontier.

The slope of the level curve of $F()$ is called the marginal rate of transformation.

Single output

Suppose we know a priori which good is the output

This is not always totally obvious

- sometimes prices determine inputs and outputs
- think about the massive pulp/bioenergy plants

Let q denote the output and z_k the inputs.

Production function $q = f\left(z_1, ..., z_{L-1}\right)$:

$$Y = \left\{\left(-z_1, ..., -z_{L-1}, q\right) : q \leq f\left(z_1, ..., z_{L-1}\right), z_k \geq 0 \text{ for all } k\right\}.$$

Firm's Problem, Profit Maximization Problem (PMP)

$$\max_{y \in Y} \quad p \cdot y.$$

Observe: No budget constraint.
When is the problem well posed (i.e. when does it have a solution)?

- We assumed that Y is closed
- What about bounded? Not necessarily
- Even though $p \cdot y$ is continuous on Y, we are not guaranteed a solution to PMP.

Denote the value function to (PMP) by $\pi(p)$.

$\pi(p)$ is called the profit function (cf. indirect utility function).

Let $y(p)$ denote the set of optimal choices at price p.

For math properties of $\pi(p)$ and $y(p)$, use Berge's Theorem of the Maximum

There is a complete duality between $\pi(p)$ and Y.

In other words: there is a one-to-one mapping between $\pi(p)$ and convex Y: A convex set is the intersection of all half-spaces containing it. (What if Y is not convex?)

Proposition (Properties of $\pi(p)$)

Assume that Y is closed and satisfies the free disposal property. Then,

1. $\pi(p)$ *is homogenous of degree one.*
2. $\pi(p)$ *is convex.*
3. *If Y is convex, then* $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \in \mathbb{R}^L_{++}\}$.
4. $y(p)$ *is homogenous of degree zero.*
5. *If Y is convex, then $y(p)$ is convex valued. If Y is strictly convex then $y(p)$ is either empty or single valued.*
6. *If $y(p)$ is single valued at p, then $\pi(p)$ is differentiable at p and $\nabla\pi(p) = y(p)$. (Hotelling's lemma).*
7. *If $y(p)$ is a function and differentiable at p, then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite with $Dy(p)p = 0$*

From properties 2 and 6 we get immediately:

$$\frac{\partial y_i}{\partial p_i} \geq 0.$$

Interpretation: If the price of an output increases, then the supply increases.
Hence we get 'Law of Supply' very easily.
Also: If the price of an input increases, the demand for the input decreases.
'Law of Input Demand'.

In the case of a single output, we have

$$\max_{z \in \mathbb{R}_+^K} \; pf(z) - w \cdot z.$$

the first order conditions are:

$$\begin{aligned} \frac{\partial f(z)}{\partial z_k} &\leq \frac{w_k}{p}, \text{ for all } k \\ \frac{\partial f(z)}{\partial z_k} &= \frac{w_k}{p} \text{ if } z_k > 0. \end{aligned}$$

Revealed Profit Approach

For any $y, y' \in Y$, we know that if $y(p) = y$ and $y(p') = y'$ then

$$p \cdot y \geq p \cdot y' \text{ and}$$

$$p' \cdot y' \geq p' \cdot y.$$

Let

$$\Delta p = \left(p' - p\right) \text{ and } \Delta y = \left(y' - y\right).$$

Then the inequalities can be written as:

$$-p \cdot \Delta y \geq 0 \text{ and } p' \cdot \Delta y \geq 0.$$

Summing these two inequalities gives the Law of Supply:

$$\Delta p \cdot \Delta y \geq 0$$

Cost Minimization

Assume a single output.
For each quantity of output, q, find the least cost input combination that yields q.
Denote input prices by $w_k > 0$.
The problem is then to

$$\min_{z \in \mathbb{R}_+^K} \quad w \cdot z$$
$$\text{s.t. } q = f\left(z_1, ..., z_K\right).$$

The solutions to this problem are $z\left(w, q\right)$, the conditional factor demands.
The value function is the cost function, $c\left(w, q\right)$

$$c\left(w, q\right) = w \cdot z\left(w, q\right).$$

Clearly $z\left(w, q\right)$ is completely analogous to $h\left(p, u\right)$ in consumer theory and $c\left(w, q\right)$ is analogous to $e\left(p, u\right)$.

Proposition (Properties of $c(w,q)$)

Assume a single output and that Y is closed and satisfies the free disposal property. Then,

1. $c()$ is homogenous of degree 0 in w and nondecreasing in q
2. $c()$ is concave in w
3. if $\{z \geq 0 : f(z) \geq q\}$ is convex $\forall q$, then $Y = \{(-z, q) : w \cdot z \geq c(w, q), \forall w \in \mathbb{R}^{L-1}_{++}\}$
4. $z(w,q)$ is homogenous of degree 0 in w

5. if $\{z \geq 0 : f(z) \geq q\}$ is convex , then the conditional factor demand $z(w,q)$ is convex-valued; if $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w,q)$ is single-valued, i.e. a function
6. if $z(w,q)$ is a function, then $z(w,q)$ is differentiable at w and satisfies $\nabla_w c(w,q) = z(w,q)$
7. if $z(w,q)$ is differentiable at w, then $D_w z(w,q) = D_w^2 c(w,q)$ is symmetric and negative semidefinite with $D_w z(w,q)\, w = 0$
8. if $f()$ is homogenous of degree 1, then $c()$ and $z()$ are homogenous of degree 1 in q
9. if $f()$ is concave, then $c()$ is convex in q.

Second Step:

Choose the optimal level of production.

$$\max_{q \in \mathbb{R}} \; pq - c\left(w, q\right).$$

Hence we get the FOC:

$$p = \frac{\partial c\left(w, q\right)}{\partial q}.$$

For competitive firms, marginal cost equals price.
Recall: $\frac{\partial c(w,q)}{\partial q} = \lambda$ for the cost minimization problem.

Differences between Consumer and Producer Theory:

- Preference representation u is unique only up to increasing transformations.
  - Production function f is a unique description of technology.
  - Hence properties such as concavity of f have a real meaning on the producer side.
- On the other hand, who is the firm?
  - Individuals decide, not abstract organizations
  - Owners decide or manager decides?
  - What if the owners do not have same preferences?

Describing technologies:

- Marginal rate of substitution:

$$MRTS_{ij} = \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}}.$$

Slope of the Isoquant $\left\{z' \in \mathbb{R}_+^K : f\left(z'\right) = q\right\}$ at z.

- Elasticity of substitution:

$$\sigma_{ij}\left(z\right) = \frac{d \ln\left(\frac{z_j}{z_i}\right)}{d \ln\left(\frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}}\right)} = \frac{d\left(\frac{z_j}{z_i}\right)}{\left(\frac{z_j}{z_i}\right)} \frac{\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j}}{d\left(\frac{\partial f(z)}{\partial z_i} / \frac{\partial f(z)}{\partial z_j}\right)}.$$

- Elasticity of scale:

$$\mu\left(z\right) = \left[\frac{d\left[\ln f\left(tx\right)\right]}{d \ln\left(t\right)}\right]_{t=1} = \frac{\Sigma_{k=1}^K z_k \frac{\partial f(z)}{\partial z_k}}{f\left(z\right)}.$$

Some consequences of the model:

Proposition

Let f be a production function that is continuous, strictly increasing and strictly quasiconcave with $f(0) = 0$. If f is homogenous of degree 1, then it is concave.

proof Consider $z^1 \gg 0$ and $z^2 z^1 \gg 0$ with $q^1 = f\left(z^1\right) > 0$ and $q^2 = f\left(z^2\right) > 0$. Homogeneity of degree 1 implies that

$$f\left(\frac{z^1}{q^1}\right) = f\left(\frac{z^2}{q^2}\right) = 1.$$

Quasiconcavity then implies that

$$f\left(\frac{z^1}{q^1+q^2} + \frac{z^2}{q^1+q^2}\right) \geq 1.$$

Homogeneity of degree 1 then implies

$$f\left(z^1+z^2\right) \geq q^1+q^2 = f\left(z^1\right) + f\left(z^2\right).$$

Extend this property to all $z \in \mathbb{R}_+^K$ by continuity.

But then for all $\lambda \in [0, 1]$, we have

$$f\left(\lambda z^1 + (1-\lambda) z^2\right) \geq f\left(\lambda z^1\right) + f\left((1-\lambda) z^2\right)$$

$$= \lambda f\left(z^1\right) + (1-\lambda) f\left(z^2\right),$$

where the last equality follows from homogeneity of degree 1. Hence f is concave. **QED**

Proposition

Let f be a production function that is continuous, strictly increasing and strictly quasiconcave with $f(0) = 0$ with the property that for all $q > 0$, there is a z such that $q = f(z)$ (i.e. f is a function onto $\mathbb{R}_+$). If f is homothetic then $c(w, .q) = h(q)\, c(w, 1)$.

Proof.

Since f is homothetic, it can be written as $f(z) = v(g(z))$ where v is strictly increasing and g is homogenous of degree 1. Since $g(0) = 0$ and v is strictly increasing, $v^{-1}(q) > 0$ for all $q > 0$. Thus for any $q > 0$, let $t = \frac{v^{-1}(1)}{v^{-1}(q)}$. Then $f(z) \geq q \;\Leftrightarrow\; v(g(tz)) \geq 1$. Thus

$$\begin{aligned}
c(w,q) &= \min_{z \in \mathbb{R}_+^K} \; w \cdot z \;\text{ s.t. }\; f(z) \geq q \\
&= \min_{z \in \mathbb{R}_+^K} \; w \cdot z \;\text{ s.t. }\; v(g(tz)) \geq 1 \\
&= \frac{1}{t} \min_{z \in \mathbb{R}_+^K} \; w \cdot tz \;\text{ s.t. }\; v(g(tz)) \geq 1 \\
&= \frac{1}{t} \min_{z' \in \mathbb{R}_+^K} \; w \cdot z' \;\text{ s.t. }\; f(z') \geq 1 \\
&= \frac{v^{-1}(q)}{v^{-1}(1)} c(w,1).
\end{aligned}$$

∎

Proposition

Suppose that $c(w, q)$ is convex in q. Then a proportional increase in all factor prices leads to a reduced revenue.

Discussion

Since the price of the output does not change, this means that output falls.

Is this true for all factor price increases?

proof of the proposition

We'll assume interior solutions throughout. First order condition for profit maximization:

$$p = \frac{\partial c\left(w, q\right)}{\partial q}.$$

A proportional price change is given by multiplying w by a constant t. Then we have:

$$p = \frac{\partial c\left(tw, q\left(t\right)\right)}{\partial q} \text{ for all } t.$$

Differentiate both sides w.r.t. t and evaluate at $t = 1$ to get:

$$0 = \frac{\partial^2 c\left(w, q\right)}{\partial q^2} q'\left(1\right) + \Sigma_{k=1}^{K} \frac{\partial^2 c\left(w, q\right)}{\partial q \partial w_k} w_k.$$

Now $c\left(w, q\right)$ is homogenous of degree 1 in w and therefore so is $\frac{\partial c(w,q)}{\partial q}$.

As a result,

$$\Sigma_{k=1}^{K} \frac{\partial^2 c\left(w,q\right)}{\partial q \partial w_k} w_k = \frac{\partial c\left(w,q\right)}{\partial q}$$

and we have:

$$q'\left(1\right) = -\frac{\frac{\partial c(w,q)}{\partial q}}{\frac{\partial^2 c(w,q)}{\partial q^2}} < 0$$

by convexity of $c\left(w,q\right)$ in q. **QED**

Aggregate Supply

Since there are only substitution effects along the production frontier, the aggregation theory for the supply side is simple (unfortunately this is not so for consumer side)

$Y_{1,}, ..., Y_J$, collection of production sets

$\pi_j(p)$, $y_j(p)$, profits and supply correspondences, $j = 1, ..., J$

the aggregate supply

$$y(p) = \sum_j y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in y_j(p), j = 1, .., J$$

The properties of $y_j(p)$ are preserved under addition. In particular, $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite.

The Law of (aggregate) Supply follows:

$$\Delta p \cdot \Delta y \geq 0$$

Let Y be the aggregate production set:

$$Y = Y_1 + ... + Y_J = \{y \in \mathbb{R}^L : y = \sum_j y_j, \text{ for some } y_j \in Y_j, j = 1, .., J\}$$

Let $\pi^* (p)$, $y^* (p)$ be the corresponding aggregate profit function and aggregate supply correspondences.

Proposition

For all $p \in \mathbb{R}^L_{++}$,

1. $\pi^* (p) = \sum_j \pi_j (p)$,
2. $y^* (p) = \sum_j y_j (p)$.