Microeconomic Theory I: Lecture 9

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Space of lotteries

Choice space: the space of lotteries.

- Set C of possible consequences
- Assume for now that C has finitely many elements $c_n \in \{c_i, ..., c_N\}$.
- A simple lottery is a vector $(p_1, ..., p_N)$ with $p_n = P(c_n)$ satisfying
  1. $p_n \geq 0$ for all n.
  2. $\sum_n p_n = 1$.
- The set $\mathcal{L}$ denotes the set of all simple lotteries:

$$\mathcal{L} = \{p \in \mathbb{R}_+^N | \sum_n p_n = 1\}.$$

- We denote by δ_{c_n} the degenerate lottery that puts probability 1 on c_n.

Our objective is to find good assumptions on preference relations $\succeq$ on $\mathcal{L}$ and a utility representation for the preferences

Compound lotteries

- Since $\mathcal{L}$ is a convex set, it makes sense to talk about *compound lotteries*, i.e. lotteries on lotteries.
- Take $L, L' \in \mathcal{L}$. A compound lottery $L^{\alpha} \in \mathcal{L}$ is a lottery on $\{L, L'\}$.
$$L^{\alpha} = \alpha L + (1 - \alpha) L'.$$
- If p_n give the probabilities of c_n under L, and p'_n give the probabilities under L', then the induced probabilities on c_n in the two-stage lottery $\mathcal{L}^{\alpha}$ are given by $(\alpha p_1 + (1 - \alpha p'_1), ..., \alpha p_N + (1 - \alpha) p'_N)$.
- We adopt here the consequentialist viewpoint that any two-stage lottery (or any multi-stage lottery) is equivalent to a simple lottery on C if they induce the same total probability distribution on C.
- Hence we can restrict attention to preferences on simple lotteries.

Preferences on lotteries

We maintain that as in choice under certainty, $\succeq$ is complete and transitive.

Axiom 1 (Rational Preference)

$\succeq$ *is a rational preference relation on* $\mathcal{L}$.

We also require that $\succeq$ be continuous and we formulate this in a manner that is traditional for choice under uncertainty, but equivalent to our previous notion.

Axiom 2 (Archimedean Axiom)

Take $L, L', L'' \in \mathcal{L}$ *such that* $L \succ L' \succ L''$. *Then the sets* $\{\alpha \in [0,1] : \alpha L + (1-\alpha) L'' \succeq L'\}$ *and* $\{\alpha \in [0,1] : \alpha L + (1-\alpha) L'' \preceq L'\}$ *are closed. for all* $\alpha \in [0,1]$ *and for all* $L'' \in \mathcal{L}$.

Can you interpret this axiom? Can you have infinitely bad or painful consequences? Why 'Archimedean'?

Independence Axiom

- The results in the first part of the lectures imply that with these axioms on preferences, a continuous utility representation exists.
- We have not used the fact that we are dealing with choice under uncertainty.
- The Independence axiom distinguishes choice under uncertainty from general choice theory

Axiom 3 (Independence Axiom)

For any $L, L' \in \mathcal{L}$, we have

$$L \succeq L' \Leftrightarrow \alpha L + (1 - \alpha) L'' \succeq \alpha L' + (1 - \alpha) L''$$

for all $\alpha \in [0, 1]$ and for all $L'' \in \mathcal{L}$.

Independence Axiom: interpretation

- In a compound lottery, you end up with one of two possible simple lotteries. If you keep one of these simple lotteries (L'') unchanged but change the other to a worse one, the second compound lottery should be worse than the first.
- This relies crucially on the probabilistic nature of the problem. In those cases where you end up with the same lottery, you should be indifferent. Hence your preference should depend only on your evaluation of the part that differs.
- Obviously, there is no equivalent to this in the theory of choice under certainty.
- Should you think of this as a positive or a normative axiom?

Expected Utility Theorem

Theorem 1

(Expected Utility Theorem) A rational preference $\succeq$ on $\mathcal{L}$ satisfies the Archimedean and Independence Axioms if and only if there exists a utility function $u : C \to \mathbb{R}$ such that

$$L \succeq L' \ \Leftrightarrow \Sigma_{n=1}^{N} p_n u_n \geq \Sigma_{n=1}^{N} p'_n u_n .$$

Furthermore, if u and u' are such representations, then $u' = \beta u + \gamma$ where $\beta > 0$.

Expected Utility Theorem: comments

- The claim is that the axioms imply a representation that is linear in the probabilities defining the lotteries.
- We call $U(L)$ the von Neumann - Morgenstern utility of lottery L.
- We call $u(c)$ the Bernoulli utility from consequence c.
- The theorem states that the von Neumann - Morgenstern utility of a lottery is computed by taking the expected value of the Bernoulli utilities on consequences, hence the name Expected Utility Theorem.
- The theorem implies a huge reduction in complexity when evaluating the lotteries.

Expected Utility Theorem: Proof

i) We show that the axioms imply the existence of such a representation.

Consider first δ_{c_n}. Since C is finite, there exist c_w and c_b such that $\delta_{c_w} \preceq \delta_{c_n}$ for all n and $\delta_{c_n} \preceq \delta_{c_b}$ for all n.

Claim 1

By independence Axiom, $\delta_{c_w} \preceq L$ for all $L \in \mathcal{L}$ and $\delta_{c_b} \succeq L$ for all $L \in \mathcal{L}$.

(Prove this as an exercise)

Expected Utility Theorem: Proof

If $\delta_{c_w} \sim \delta_{c_b}$, we can take u to be any constant function and the theorem is proved. Assume thus that $\delta_{c_w} \prec \delta_{c_b}$. Let $u(\delta_{c_w}) = 0$ and $u(\delta_{c_b}) = 1$.

Claim 2

Since $\delta_{c_w} \prec \delta_{c_b}$, Independence Axiom implies that

$$p\delta_{c_b} + (1-p)\,\delta_{c_w} \succ q\delta_{c_b} + (1-q)\,\delta_{c_w} \text{ iff } p > q.$$

(Prove this as an exercise)

Expected Utility Theorem: Proof

Claim 3

For each $L \in \mathcal{L}$, there is a unique $\alpha(L)$ such that $L \sim \alpha(L)\,\delta_{c_b} + (1-\alpha(L))\,\delta_{c_w}$.

Proof.

By Archimedean axiom, $\{\alpha : \alpha\delta_{c_b} + (1-\alpha)\,\delta_{c_w} \succeq L\}$ and $\{\alpha : \alpha\delta_{c_b} + (1-\alpha)\,\delta_{c_w} \preceq L\}$ are closed. By the completeness of $\succeq$, the union of these sets is [0,1]. Since $[0,1]$ is a connected, the intersection of the sets must be nonempty. By Claim 2, the intersection must be a singleton.

We write $\alpha(L)$ for the unique element in the intersection. ∎

Expected Utility Theorem: Proof

Claim 4

For all L, L' and $\gamma \in [0,1]$ we have

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right).$$

Proof:

$$\gamma L + (1-\gamma) L'$$

$$\sim \gamma[\alpha(L)\delta_{c_b} + (1-\alpha(L))\delta_{c_w}] + (1-\gamma)\left[\alpha\left(L'\right)\delta_{c_b} + \left(1-\alpha\left(L'\right)\right)\delta_{c_w}\right]$$

$$\sim [\gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right)]\delta_{c_b} + [\gamma(1-\alpha(L)) + (1-\gamma)\left(1-\alpha\left(L'\right)\right)]\delta_{c_w}$$

And thus

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right). QED.$$

This shows that $\alpha(L)$ is linear on $\mathcal{L}$ and therefore $\alpha(L) = \Sigma_{n=1}^{N} p_n u_n$ for some $u : C \to \mathbb{R}$ and we can take $U(L) = \alpha(L)$ and $u_n = \alpha(\delta_{c_n})$.

Expected Utility Theorem: Proof

ii) It is left as an exercise to verify that any representation of this form implies that the underlying preferences satisfy Axiom1-3.
iii) If u' and u'' are the Bernoulli utility functions, let U' and U'' be the corresponding von Neumann-Morgenstern utility functions. The claim is proved if we prove it for U' and U''.
As before, let $\alpha'(L)$ solve

$$U'(L) = \alpha'(L)\, U'(\delta_{c_b}) + \left(1 - \alpha'(L)\right) U'(\delta_{c_w}).$$

Thus

$$\alpha'(L) = \frac{U'(L) - U'(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}.$$

Expected Utility Theorem: Proof

But now since U'' is also a representation, we have

$$U''(L) = \alpha'(L)\, U''(\delta_{c_b}) + \left(1 - \alpha'(L)\right) U''(\delta_{c_w}).$$

Plugging in the value of $\alpha'(L)$ and rearranging, we get:

$$U''(L) = \beta U'(L) + \gamma,$$

where

$$\beta = \frac{U''(\delta_{c_b}) - U''(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}$$

and

$$\gamma = U''(\delta_{c_w}) - U'(\delta_{c_w}) \frac{U''(\delta_{c_b}) - U''(\delta_{c_w})}{U'(\delta_{c_b}) - U'(\delta_{c_w})}.$$

Expected Utility Theorem: Scope of applications

- Above, we take the probabilities to be objectively given.
- In a much more complicated setting, the same expected utility formula can be derived for subjective probability assessments.
- For decision theory, it is often useful to write the consequences as $C := \Omega \times A$, where Ω is an exogenous probability space, called states of nature and A is the set of possible actions. Then

$$U(a) = \sum_{\omega \in \Omega} u(\omega, a) p(\omega).$$

- For example, $\omega \in \{\omega^H, \omega^L\}$ parametrizes the demand function for a firm's product for next year. $a \in \mathbb{R}_+$ is the investment decision by the firm and $u(\omega, a)$ is the revenue in state ω at investment level a.
- Statistical decision making: $\Omega = \{true, false\}$ and $A = \{accept, reject\}$ and $u(\omega, a)$ are the costs of the decisions.

Preference on Monetary Payoffs

By far the most analyzed special case of choice under uncertainty is the case where uncertainty is about monetary consequences.

- $x \in \mathbb{R}$ is the final wealth of the decision maker.
- Random variable X captures the uncertainty.
- Analyze different $u : \mathbb{R} \to \mathbb{R}$.
- Let $F(x)$ denote the distribution function (discrete or continuous) of a lottery on final wealths.

Preference on Monetary Payoffs

Expected Utility Theorem:

$$U(F) = \int_{x \in \mathbb{R}} u(x) dF(x),$$

where $F(x)$ is the c.d.f. of X.

Or for discrete distributions:

$$U(F) = \sum_{x} u(x) p(x),$$

where $p(x)$ is the mass function of the lottery.

Note carefully that we take the expectation of the utility, NOT the utility of expectation.

Preferences on Monetary Payoffs

- When do the integrals and sums above converge?
- For sure if X takes has only finitely many possible values.
- For sure if u is bounded.
- But we'd really like to be able to handle, say normally or log-normally distributed final wealths and Bernoulli utility functions that are exponential or power functions (you'll see in a bit why these are important).
- Just make sufficient assumptions of the tails of the variables and growth rates of the utility functions and you'll be OK.

Preferences on Monetary Payoffs

- Note the symmetry in the formula between $u(x)$ and $F(x)$.
- We will consider variations in each of these two components.
  - Risk attitudes: Fix $F(x)$ and compare different $u(x)$.
  - Riskiness of lotteries: Fix $u(x)$ and compare different $F(x)$.
- In this lecture, comparisons of risk attitudes, in the next, comparisons of risks.

Risk Attitudes

Definition The certainty equivalent $c(F,u)$ of a lottery $F(X)$ for a decision maker with (Bernoulli) utility function defined by

$$u(c(F,u)) = \int u(x)dF(x)$$

We can discuss attitudes towards risk by comparing the certainty equivalents of a fixed lottery under different utility functions.

Definition A decision maker with a utility function u is said to be risk averse if for all $F(x)$,

$$c(F,u) \leq \int xdF(x).$$

Risk Aversion

It is easy to prove that

Proposition Utility function u is risk averse if and only if it is concave.

Risk loving attitudes are defined with the opposite inequalities.

Can second derivatives be used to measure risk aversion?
Definition The Arrow-Pratt measure of absolute risk aversion, $r_A(x, u)$ of utility function u at wealth level x is given by:

$$r_A(x, u) = -\frac{u''(x)}{u'(x)}.$$

Absolute Risk Aversion

The following theorem shows that $r_A(x, u)$ is a good measure of risk aversion.

Proposition The following are equivalent:
i) $r_A(x, u_2) \geq r_A(x, u_1)$ for all x.
ii) $c(F, u_2) \leq c(F, u_1)$ for all $F(x)$.
iii) There is a concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$.

Relative Risk Aversion

A related concept is the measure of relative risk aversion:

$$r^R(x, u) = -\frac{xu''(x)}{u'(x)}.$$

$r^R(x, u)$ measures the attitudes towards gambles proportional to wealth.

(Weak) Empirical evidence suggests:

People are risk averse.

Absolute risk aversion decreases with wealth.

Relative risk aversion decreases or is constant with wealth.

Prudence

Define the prudence of a Bernoulli utility function u as:

$$P(x, u) = -\frac{u'''(x)}{u''(x)}.$$

In other words, $P(x, u)$ is the risk aversion of $-u'(x)$. Why are we interested in the negative of $u(x)$ rather than $u(x)$ itself?

What are the conditions for decreasing absolute risk aversion and decreasing relative risk aversion expressed with $P(x, u)$ and $r^A(x, u)$?

CARA and CRRA Bernoulli functions

- Let X be a lottery on final wealth with distribution F, and let $w+X$ denote the lottery induced by adding an initial wealth w to X. Let $F^w(x) = \Pr\{w+X \leq x\}$.
- We say that the decision maker's risk attitudes are independent of initial wealth if for all F and all w,

$$c(F,u) = c(F^w,u).$$

- in this case, the Bernoulli utility function has constant absolute risk aversion (CARA):

$$u(x) = -e^{-\gamma x},$$

and $r^A(x,u) = \gamma$ for all x.

CARA and CRRA Bernoulli functions

- We could also ask about the attitudes towards proportional risks, i.e. for any initial wealth x, the final wealth is tx and t is a random variable with distribution function $F(t)$.
- If the decision maker's evaluation of different $F(t)$ is independent of x, we say that the decision maker has constant relative risk aversion (CRRA).
- If

  $$u(x) = \frac{x^{1-\rho}}{1-\rho} \text{ for } \rho \neq 1,$$

  then $r^R(s,u) = \rho$ for all x. It is left as an exercise to show that (CRRA) implies the above form for the Bernoulli utility function.
- Show that in the limit as $\rho \to 1$, we get $u(x) = \ln x$.

Example: Simple betting

- A decision maker with a strictly concave Bernoulli utility function u with $u''(\cdot) < 0$.
- Initial wealth w.
- Two possible states of the world $\{\omega_0, \omega_1\}$.
- $\Pr\{\omega = \omega_0\} = p$.
- A bet of size $x \geq 0$ costs qx pays off x if $\omega = \omega_0$ and zero otherwise.
- Final wealth as a function of x: $w + x - qx$ with probability p and $w - qx$ with probability $(1 - p)$.

Example: Simple betting

- Expected utility from bet x:

$$U(x) = pu(w + x - qx) + (1 - p)u(w - qx).$$

- The concavity of u implies the concavity of U
- The bet is fair if $p = q$, it is favorable if $p > q$, and it is unfavorable if $p < q$.
- Condition for interior maximum:

$$\frac{p(1-q)}{(1-p)q} = \frac{u'(w - qx)}{u'(w + x - qx)}.$$

- The decision maker chooses $x^* = 0$ for $p \leq q$ and $x^* > 0$ for $p > q$
- All such decision makers bet strictly positive amounts. Can you find concave decision makers where this is not true? Is there something specila about such DM's?

Example: Insurance

- Agent has initial wealth w but runs a risk of losing $d > 0$ with probability p.
- Insurance costs q euros per one euro loss covered.
- When α units of coverage bought, final wealth is $(w - d - \alpha q + \alpha)$ with probability p and $(w - \alpha q)$ with probability $(1 - p)$.
- Expected utility as a function of coverage

$$U(\alpha) = pu(w - d - \alpha q + \alpha) + (1 - p)u(w - \alpha q)\}.$$

- Since u is assumed to be strictly concave, $U''(\alpha) < 0$.
- Hence first order condition for optimal α is also sufficient and the optimal α^* must satisfy:

$$-(1 - p)qu'(w - \alpha q) + p(1 - q)u'(w - d - \alpha q + \alpha) \leq 0$$

- If $q = p$, then $\alpha^* = d$. That is, the agent insures fully.

Example: diversification

Example

- A strictly risk averse decision maker must allocate an amount y between two identical independent assets.
- Denote the random return on asset i by $(1 + r_i)$ and let $f(r_i)$ denote the density function of the return.
- Let α be the fraction of wealth invested in opportunity 1.
- The final wealth of the investor is then

$$\widetilde{w}(\alpha) = (1 + r_1)\alpha y + (1 + r_2)(1 - \alpha) y.$$

Diversification continued

- The expected utility of the investor that follows strategy α is then (by independence):

$$v(\alpha) =$$

$$\int \int u\left(\left(1 + r_1\right)\alpha y + \left(1 + r_2\right)\left(1 - \alpha\right) y\right) f\left(r_1\right) f\left(r_2\right) dr_1 dr_2.$$

- Notice that

$$v''(\alpha) =$$

$$\int \int \left(r_1 - r_2\right)^2 y^2 u''\left(\left(1 + r_1\right)\alpha y + \left(1 + r_2\right)\left(1 - \alpha\right) y\right) f\left(r_1\right) f\left(r_2\right) dr$$

since the decision maker is strictly risk averse.

- By concavity, first order conditions are also sufficient for maximum.

Diversification continued

- The optimal α is found by setting

$$0 = v'(\alpha) =$$

$$\int\int (r_1 - r_2)\, yu'\left((1+r_1)\,\alpha y + (1+r_2)(1-\alpha)\, y\right) f(r_1)\, f(r_2)\, dr_1 d$$

.

- But then we must have

$$\int\int r_1 u'\left((1+r_1)\,\alpha y + (1+r_2)(1-\alpha)\, y\right) f(r_1)\, f(r_2)\, dr_1 dr_2 =$$

$$\int\int r_2 u'\left((1+r_1)\,\alpha y + (1+r_2)(1-\alpha)\, y\right) f(r_1)\, f(r_2)\, dr_1 dr_2.$$

- Since $f(r_1) = f(r_2)$, the two sides are equal if $\alpha = \frac{1}{2}$.
- By strict concavity, this must be the only solution.