Microeconomic Theory 1
Helsinki GSE – Fall 2019
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Problem Set 2 - Solutions

Problem 1

(a) **True.** Suppose $x_n \to x$, $y_n \to y$, and $x_n \succeq y_n$ for all n. Since u is a utility representation, $u(x_n) \geq u(y_n)$ for all n. Since u is continuous, we also have $u(x_n) \to u(x)$ and $u(y_n) \to u(y)$. Combining these two facts, we can conclude that $u(x) \geq u(y)$, which is equivalent to $x \succeq y$.

(b) **True.** Take $X = [0, 1]$ and suppose $x \succeq y$ if and only if $x \geq y$. Clearly, this preference relation is continuous. A valid utility representation is the discontinuous function u such that $u(x) = x$ if $x \leq \frac{1}{2}$, and $u(x) = 1 + x$ if $x > \frac{1}{2}$.

(c) **False.** For every x, we have

$$\begin{aligned} W(x) &:= \{y : x \succeq y\} \\ &= \{y : u(x) \geq u(y)\} \\ &= (-\infty, \lfloor x \rfloor + 1), \end{aligned}$$

which is not closed. This is sufficient to establish that preferences are not continuous.

(d) **False.** Let preferences $\succeq$ and utility u be the same as in part (b). In addition, let $v(x) = x$. Clearly, v represents the same preferences as u. Now, any $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $v(x) = f(u(x))$ must satisfy the following:

$$f(y) = \begin{cases} y, & \text{if } 0 \leq y \leq \frac{1}{2} \\ y - 1, & \text{if } \frac{3}{2} < y \leq 2. \end{cases}$$

Since the domain of f is the entire real line, we have that, for every value that f can possibly take on in the interval $\left(\frac{1}{2}, \frac{3}{2}\right]$, strict monotonicity cannot hold.

The counterexample above crucially depends on the requirement that f be defined over the entire real line. If we require f to be defined on the range of u, then the statement is **true**. To see this, let $U(x)$ be the range of $u(x)$. For any $u' \in U(x)$, let $g(u') := \{x \in X : u(x) = u'\}$ be the inverse image of utility level u' under u. Since v represents the same preferences as u, it must be the case that $v(x)$ is constant for every $x \in g(u')$. This implies that $f(u') = v(g(u'))$ is a well-defined function. And since u and v are equivalent representations, it follows that f is strictly increasing.

Problem 2

(a) The utility representation is $u(x_1, x_2) = a_1x_1 + a_2x_2$ for some constants $a_1, a_2 > 0$.

- **Linearity.** Suppose $(x_1, x_2) \succeq (y_1, y_2)$. Then we have:

$$\begin{aligned}(x_1, x_2) \succeq (y_1, y_2) &\iff a_1x_1 + a_2x_2 \geq a_1y_1 + a_2y_2 \\ &\iff a_1x_1 + a_1t + a_2x_2 + a_2s \geq a_1y_1 + a_1t + a_2y_2 + a_2s \\ &\iff a_1(x_1 + t) + a_2(x_2 + s) \geq a_1(y_1 + t) + a_2(y_2 + s) \\ &\iff (x_1 + t, x_2 + s) \succeq (y_1 + t, y_2 + s).\end{aligned}$$

- **Monotonicity.** Let $(x_1, x_2), (y_1, y_2)$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. This implies $a_1x_1 + a_2x_2 \geq a_1y_1 + a_2y_2$, which is equivalent to $(x_1, x_2) \succeq (y_1, y_2)$. If, in addition, either $x_1 > y_1$ or $x_2 > y_2$, then it is immediate that $a_1x_1 + a_2x_2 > a_1y_1 + a_2y_2$, which is equivalent to $(x_1, x_2) \succ (y_1, y_2)$.
- **Continuity.** The utility function u is linear, hence continuous. Thus we can use the claim from part (a) in Problem 1 to conclude that the preferences represented by u are continuous.

(b)
- **i) and ii) but not iii).** Lexicographic preferences over $\mathbb{R}^2_+$ defined as $(x_1, x_2) \succeq (y_1, y_2)$ if either $[x_1 > y_1]$ or $[x_1 = y_1$ and $x_2 \geq y_2]$.
- **i) and iii) but not ii).** Preferences over $\mathbb{R}^2_+$ represented by $u(x_1, x_2) = a_1x_1 + a_2x_2$ for some constants $a_1 > 0$ and $a_2 < 0$. In other words, commodity 2 is a "bad".
- **ii) and iii) but not i).** Preferences over $\mathbb{R}^2_+$ represented by the utility function $u(x_1, x_2) = (x_1 + 1)^\alpha (x_2 + 1)^{1-\alpha}$ for some $\alpha \in (0, 1)$. Can you see why there is "+1" in the utility? Or, why is $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ an incorrect answer?

(c) We already know that, since preferences are continuous, there exists a continuous representation u. We need to show that such a representation is linear. We proceed in several steps.

- *Step 1.* For all $x, y \in \mathbb{R}_+$, and for every positive rational number $\frac{n}{m}$, if $u(x, 0) \geq u(0, y)$, then $u(\frac{n}{m}x, 0) \geq u(0, \frac{n}{m}y)$. To show this, we first establish that $u(nx, 0) \geq u(0, ny)$ for every natural number n. The argument is by induction. When $n = 1$, the claim is obviously true. For the inductive step, suppose that $u(kx, 0) \geq u(0, ky)$ for some $k \in \mathbb{N}$. Since $u((k+1)x, 0) = u(kx + x, 0)$, and since $u(kx, 0) \geq u(0, ky)$ by the inductive hypothesis, it follows from additivity that $u((k+1)x, 0) \geq u(x, ky)$. Now, we also have that $u(x, 0) \geq u(0, y)$ by assumption. By additivity, it follows that $u(x, ky) \geq u(0, y + ky)$. Therefore, we have $u((k+1)x, 0) \geq u(0, (k+1)y)$ and we can conclude that the statement holds for every n.

  To show that the statement holds for every positive rational, suppose by way of contradiction that $u(x, 0) \geq u(0, y)$ but $u(\frac{n}{m}x, 0) < u(0, \frac{n}{m}y)$ for some $\frac{n}{m}$. We already proved that $u(nx, 0) \geq u(0, ny)$ for every n. But then $u(\frac{n}{m}x, 0) < u(0, \frac{n}{m}y)$ implies $u(nx, 0) < u(0, ny)$, so reaching a contradiction.

- *Step 2.* For any $x \in \mathbb{R}_+$, there exists a unique $f(x) \in \mathbb{R}_+$ such that $u(x, 0) = u(0, f(x))$. This follows from continuity and monotonicity.

- *Step 3.* For any positive rational number p, $\frac{f(p)}{p} = \alpha$ for some fixed $\alpha > 0$. Take a rational $p = \frac{n}{m}$. By the previous step, $u(p, 0) = u(0, f(p))$ and $u(1, 0) = u(0, f(1))$. By Step 1, the latter equality implies $u(p1, 0) = u(0, pf(1))$. Thus we have $\frac{f(p)}{p} = f(1) =: \alpha$.

- *Step 4.* For any $x \in \mathbb{R}_+$ we have $f(x) = \alpha x$. Fix $x > 0$ and take a sequence q_n of strictly positive rational numbers that converge to x. By Steps 2 and 3, $u(q_n, 0) = u(0, \alpha q_n)$. By continuity, in the limit we have $u(x, 0) = u(0, \alpha x)$.

- *Step 5.* In the final step, it remains to show that $u(x_1, x_2) = \alpha x_1 + x_2$ is indeed a utility representation. When $x \geq y$ or $y \geq x$, then the claim follows easily from monotonicity. Suppose $y_1 < x_1$ and $x_2 < y_2$ and $x \succeq y$. By additivity, $x \succeq y$ is equivalent to $(x_1 - y_1, 0) \succeq (0, y_2 - x_2)$. By Steps 3 and 4, $(x_1 - y_1, 0) \sim (0, \alpha(x_1 - y_1))$, so that $x \succeq y$ is equivalent to $(0, \alpha(x_1 - y_1)) \succeq (0, y_2 - x_2)$. By monotonicity, the latter is equivalent to $\alpha(x_1 - y_1) \geq y_2 - x_2$, which is $u(x) \geq u(y)$. The argument for the remaining case with $y_1 > x_1$ and $x_2 > y_2$ is analogous.

Problem 3

(a) We start by observing that

$$\left(p^1, I^1\right) = \frac{1}{2}\left(p^2, I^2\right) + \frac{1}{2}\left(p^3, I^3\right). \tag{1}$$

Recall that the utility level for a chosen bundle is given by the indirect utility function $v(p, I)$. Since v is a quasi-convex function, we can use (1) to write

$$v\left(p^1, I^1\right) \leq v\left(p^2, I^2\right) = v\left(p^3, I^3\right).$$

(b) Let $\lambda = \frac{1}{5.94573}$ indicate the exchange rate. We have

$$\left(p^4, I^4\right) = \lambda\left(p^3, I^3\right).$$

Since the Walrasian demand is homogeneous of degree zero, it follows that $x\left(p^4, I^4\right) = x\left(p^3, I^3\right)$ and, consequently, the utility attained at $t = 4$ is exactly the same as in $t = 3$.

(c) Suppose that the price of good i increases in $t = 5$, i.e. $p_i^5 > p_i^4$. Define $u^4 := v(p^4, I^4)$. Recall that the Hicksian and Walrasian demands must satisfy

$$x\left(p^4, I^4\right) = h\left(p^4, u^4\right)$$

At $t = 5$, the chosen bundle is such that the utility level attained is the same as in $t = 4$. In other words, this means that

$$x\left(p^5, I^5\right) = h\left(p^5, u^4\right),$$

where $I^5 = e\left(p^5, u^4\right) = p^5 \cdot h\left(p^5, u^4\right)$. Since the Hicksian demand satisfies the compensated law of demand, we have

$$\left(p_i^5 - p_i^4\right)\left[h_i\left(p^5, u^4\right) - h_i\left(p^4, u^4\right)\right] \leq 0,$$

from which we can easily conclude that $x_i^5 \leq x_i^4$.

Problem 4

Suppose that the consumer is maximizing a strictly quasi-concave utility function. In this case, we know that the underlying preferences are strictly convex and that the Walrasian demand is a singleton for any price-income combination.

Let $p = (1, 1, 1)$, $p' = (4, 6, 4)$, $x = (1, 2, 3)$, and $x' = (3, 2, 1)$. When prices are p, we have that $p \cdot x = p \cdot x' = 6$. Since x is chosen, and x' is affordable when x is chosen, we can conclude that $x \succ x'$. In other words, with strictly convex preferences, the chosen bundle is strictly preferred to all other affordable alternatives.

When prices are p', we have that $p' \cdot x' = p' \cdot x = 28$. Since x' is chosen, and x is affordable when x' is chosen, we can conclude that $x' \succ x$, so contradicting the previous claim that x is strictly preferred to x'. Therefore, the maximization of a strictly quasi-concave utility cannot give rise to the observed choices. However, these choices are not incompatible with quasi-concave utility.

Problem 5

The utility function is $u(x_1, \ldots, x_L) = (\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho)^{\frac{1}{\rho}}$, with $\rho < 1$, $\rho \neq 0$, and strictly positive coefficients α_i, $i = 1, \ldots, L$.

(a) For all i, the partial derivative of u with respect to x_i is

$$\alpha_i \left(\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho\right)^{\frac{1}{\rho}-1} x_i^{\rho-1}, \tag{2}$$

which is strictly positive provided that $x_i > 0$. This means that preferences are monotone, hence locally non-satiated. Therefore, Walras' law holds. In addition, (2) approaches infinity as x_i goes to zero. Therefore, for the Kuhn-Tucker conditions to hold, all x_i must be strictly positive. Another way to see that the solution must be interior is the following. Suppose x is utility maximizing and that some of its components are zero. Without loss of generality, suppose $x_i = 0$ for the first $K < L$ commodities, and strictly positive for the remaining $L - K$ goods. Take a bundle x' in the budget set such that $x_i' > 0$ for $i = 1, \ldots, K$, and $x_i' = 0$ for $i = K+1, \ldots, L$. Now consider the convex combination $\alpha x + (1-\alpha)x'$, with $\alpha \in [0, 1]$. Clearly, any such combination is feasible due to the convexity of the budget set, and it lies in the interior of the domain when $\alpha \in (0, 1)$. But then you can verify that the utility $u(\alpha x + (1-\alpha)x')$ is maximized at

$$\alpha = \frac{u(x')^{\frac{\rho}{\rho-1}}}{u(x)^{\frac{\rho}{\rho-1}} + u(x')^{\frac{\rho}{\rho-1}}},$$

which lies in $(0, 1)$. This contradicts the assumption that x, i.e. $\alpha = 1$, is a solution.

The utility maximization problem reduces to maximize $u(x)$ subject to the budget constraint, which is binding by Walras' hold. From the necessary (and sufficient) conditions for optimality we easily obtain that, for any i and j,

$$\alpha_i \left(\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho\right)^{\frac{1}{\rho}-1} x_i^{\rho-1} = \lambda p_i \tag{3}$$

$$\alpha_j \left(\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho\right)^{\frac{1}{\rho}-1} x_j^{\rho-1} = \lambda p_j, \tag{4}$$

from which we can conclude that the marginal rate of substitution is equal to $\frac{p_i}{p_j}$.

(b) Dividing (3) by (4) side by side and solving for x_j yields

$$x_j = \left(\frac{p_j}{p_i}\right)^{\frac{1}{\rho-1}} \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{1}{\rho-1}} x_i. \tag{5}$$

(c) Plugging (5) in the budget constraint for any $j \neq i$ and then solving for x_i yields the expression for the Walrasian demand:

$$x_i(p,w) = \frac{w}{p_i + \sum_{j\neq i} p_j \left(\frac{p_j}{\alpha_j}\right)^{\frac{1}{\rho-1}} \left(\frac{\alpha_i}{p_i}\right)^{\frac{1}{\rho-1}}}. \tag{6}$$

If we multipy and divide the right-hand side of (6) by $\left(\frac{p_i}{\alpha_i}\right)^{\frac{1}{\rho-1}}$, we can rewrite the Walrasian demand in a more compact form:

$$x_i(p,w) = \frac{\left(\frac{p_i}{\alpha_i}\right)^{\frac{1}{\rho-1}} w}{\sum_{j=1}^{L} p_j \left(\frac{p_j}{\alpha_j}\right)^{\frac{1}{\rho-1}}}.$$

(d) When $a_i = 1$ for all i, the Walrasian demand for any i reduces to

$$x_i(p,w) = \frac{p_i^{\frac{1}{\rho-1}} w}{\sum_{j=1}^{L} p_j^{\frac{\rho}{\rho-1}}}. \tag{7}$$

The indirect utility is $v(p,w) = u(x(p,w))$. By substituting (7) in the utility function and rearranging we obtain

$$v(p,w) = \left(\sum_{j=1}^{L} p_j^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w.$$

Now, the derivative of $v(p,w)$ with respect to p_i is

$$\frac{\partial v(p,w)}{\partial p_i} = -\left(\sum_{j=1}^{L} p_j^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} p_i^{\frac{1}{\rho-1}} w. \tag{8}$$

The derivative of $v(p,w)$ with respect to w is

$$\frac{\partial v(p,w)}{\partial w} = \left(\sum_{j=1}^{L} p_j^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}. \tag{9}$$

Dividing (8) by (9) and multiplying by -1 gives exactly the Walrasian demand (7).

Problem 6

(i) Let us consider the case with two goods. For each consumer i, the individual demand is a random vector (x_1^i, x_2^i), where x_1^i is uniformly distributed over $\left[0, \frac{w}{p_1}\right]$ and $x_2^i = \frac{w}{p_2} - \frac{p_1}{p_2} x_1^i$. Therefore, the average individual demand is $\bar{x}_1^i = \frac{w}{2p_1}$ and $\bar{x}_2^i = \frac{w}{p_2} - \frac{p_1}{p_2}\bar{x}_1^i = \frac{w}{2p_2}$. Since consumers are assumed to the homogeneous, and since they all face the same price-income pair, the average market demand coincides with the average individual demand. Hence we have

$$\bar{x}(p, w) = \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right).$$

In the general case with L commodities, you can verify that the average market demand is

$$\bar{x}(p, w) = \left(\frac{w}{Lp_1}, \frac{w}{Lp_2}, \ldots, \frac{w}{Lp_L}\right).$$

(ii) Yes, the average demand does satisfy the weak axiom of revealed preference. To see this, take any two price-income pairs (p, w) and (p', w'). Suppose that $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$. We need to show that $p' \cdot x(p, w) > w'$. If $\frac{p_1}{p_2} = \frac{p_1'}{p_2'}$, it is easy to see that the claim holds. In the complementary case, observe that the hypothesis $p \cdot x(p', w') \leq w$ can be rewritten as

$$\frac{1}{2}\left(\frac{p_1}{p_1'} + \frac{p_2}{p_2'}\right) \leq \frac{w}{w'}.$$

Since $\frac{p_1}{p_2} \neq \frac{p_1'}{p_2'}$ by assumption, we have that

$$\frac{2}{\frac{p_1'}{p_1} + \frac{p_2'}{p_2}} < \frac{1}{2}\left(\frac{p_1}{p_1'} + \frac{p_2}{p_2'}\right),$$

where the left-hand side is the harmonic mean of $\frac{p_1}{p_1'}$ and $\frac{p_2}{p_2'}$ and the right-hand side is their arithmetic mean. Therefore,

$$\frac{2}{\frac{p_1'}{p_1} + \frac{p_2'}{p_2}} < \frac{w}{w'},$$

which is equivalent to $p' \cdot x(p, w) > w'$.

(iii) The observed choices can be explained by a linear utility function $u(x_1, x_2) = a_1 x_1 + a_2 x_2$ such that $\frac{a_1}{a_2} = \frac{p_1}{p_2}$. In this case, it is easy to see that every bundle on the budget line is utility maximizing.

Problem 7

(a) Given prices (p_x, p_y), income w, and tax rates t_x and t_w, let (x_c^*, y_c^*) be the demanded bundle under a commodity tax. The resulting tax revenue is $g = t_x x_c^*$. By Walras' law, we have $(p_x + t_x)x_c^* + p_y y_c^* = w$, which is equivalent to $p_x x_c^* + p_y y_c^* = w - g$. The latter says that (x_c^*, y_c^*) would be affordable even if the government raised the the tax revenue $g = t_x x_c^*$ via an income tax t_w such that $t_w w = g = t_x x_c^*$. Therefore, the consumer cannot be worse off under income taxation.

(b) The choice is between two commodity taxes. See Section 3.I of the texbook for a general discussion of this case.

Problem 8

Fix a pair (p, w) and specialize the given expression to the case when $u = v(p, w)$. We know that $e(p, v(p, w)) = w$. Using this, we have

$$w = a(p) + b(p)v(p, w),$$

which is equivalent to

$$v(p, w) = \frac{w - a(p)}{b(p)}.$$

By Roy's identity, we obtain

$$\begin{aligned} x_i(p, w) &= -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} \\ &= \frac{1}{b(p)} \left[\frac{\partial a(p)}{\partial p_i} b(p) + (w - a(p)) \frac{\partial b(p)}{\partial p_i} \right] \\ &= \frac{\partial a(p)}{\partial p_i} + \frac{\partial b(p)}{\partial p_i} \left[\frac{w - a(p)}{b(p)} \right]. \end{aligned}$$

Finally, observe that the derivative of $x_i(p, w)$ with respect to w is

$$\frac{\partial x_i(p, w)}{\partial w} = \frac{\partial b(p)}{\partial p_i} \frac{1}{b(p)},$$

which is a constant term that does not depend on w itself. This means that the share of income that is spent on any commodity remains constant as wealth changes (and prices are kept fixed). Can you find utility functions that give rise to this behavior?