Microeconomic Theory I: Lecture 10

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Comparing risks

- Classification of risks for a pre-specified class of utility functions.
- The notions of stochastic dominance will thus be different depending on the class of utility functions that are considered.
- The two most commonly used notions are first order stochastic dominance and second order stochastic dominance.
- If Ω is the set of utility functions that are of interest, we look for characterizations of distributions $F_1(x)$ and $F_2(x)$ to guarantee that

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ for all } u(x) \in \Omega. \quad (1)$$

First-Order Stochastic Dominance

- For first-order stochastic dominance, the class of functions of interest are increasing functions. We assume differentiability of the functions throughout and also that all the integrals converge (e.g. by having a bounded domain or bounded functions on an unbounded domain).

$$\Omega^1 = \left\{ u\left(x\right) : u'\left(x\right) \geq 0 \right\}.$$

Definition
A distribution $F_1\left(x\right)$ first order stochastically dominates $F_2\left(x\right)$ if

$$\int u\left(x\right) dF_1\left(x\right) \geq \int u\left(x\right) dF_2\left(x\right) \text{ for all } u\left(x\right) \in \Omega^1.$$

First-Order Stochastic Dominance

Proposition $F_1(x)$ first order stochastically dominates $F_2(x)$ if and only if

$$F_1(x) \leq F_2(x) \text{ for all } x \in [\underline{x}, \overline{x}].$$

Proof Integration by parts gives:

$$\int_{\underline{x}}^{\overline{x}} u(x)[dF_1(x) - dF_2(x)] \geq 0$$

$$\iff \Big|_{\underline{x}}^{\overline{x}} u(x)[F_1(x) - F_2(x)] - \int_{\underline{x}}^{\overline{x}} u'(x)[F_1(x) - F_2(x)]dx \geq 0.$$

The first term is zero since $F_i(\underline{x}) = 0$ and $F_i(\overline{x}) = 1$ for $i = 1, 2$.

First-Order Stochastic Dominance

i) If $F_1(x) \leq F_2(x)$ for all $x \in [\underline{x}, \overline{x}]$, then the second integral is negative for all u with $u'(x) \geq 0$ for all x. And as a result, F_1 first-order stochastically dominates F_2.

ii) Conversely, if $F_1(x') > F_2(x')$ for some x', then $x' < \underline{x}$ and since F_i are continuous from the right, $\exists \varepsilon > 0$ such that $F_1(x) > F_2(x) \forall x \in (x', x' + \varepsilon)$. Consider any u such that $u'(x) = 0$ if $x \notin (x', x' + \varepsilon)$ and $u'(x) > 0$ if $x \in (x', x' + \varepsilon)$. Then F_1 does not first-order stochastically dominate F_2. QED.

Second-Order Stochastic Dominance

- For this notion, the relevant class of Bernoulli utility functions is given by increasing concave functions:

$$\Omega^{2'} = \left\{ u(x) : u'(x) \geq 0, u''(x) \leq 0 \right\}.$$

Definition A distribution $F_1(x)$ second-order stochastically dominates $F_2(x)$ if

$$\int u(x)\, dF_1(x) \geq \int u(x)\, dF_2(x) \text{ for all } u(x) \in \Omega^2.$$

- In some cases, we do not insist on u being increasing.

Second-Order Stochastic Dominance

Proposition $F_1(x)$ second order stochastically dominates $F_2(x)$ if and only if

$$\int_{\underline{x}}^{x} F_1(y)\, dx \leq \int_{\underline{x}}^{x} F_2(y)\, dy \text{ for all } y.$$

Proof From the previous proposition, we have after integration by parts:

$$\int_{\underline{x}}^{\overline{x}} u(x)[dF_1(x) - dF_2(x)] = -\int_{\underline{x}}^{\overline{x}} u'(x)[F_1(x) - F_2(x)]dx.$$

A second integration by parts gives:

$$-\Big|_{\underline{x}}^{\overline{x}} u'(x)[\int_{\underline{x}}^{x} (F_1(y) - F_2(y))dy] + \int_{\underline{x}}^{\overline{x}} u''(x)[\int_{\underline{x}}^{x} (F_1(y) - F_2(y))dy]dx.$$

Second-Order Stochastic Dominance

The first term is equal to:

$$-u'(\overline{x}) \int_{\underline{x}}^{\overline{x}} (F_1(x) - F_2(x))dx,$$

and by another integration by parts it equals:

$$= u'(\overline{x})[\mathbb{E}_{F_1}(X) - \mathbb{E}_{F_2}(X)].$$

Hence F_1 second-order stochastically dominates F_2 if

$$\int_{\underline{x}}^{x} F_1(y)\, dy \leq \int_{\underline{x}}^{x} F_2(y)\, dy \text{ for all } x.$$

and

$$\mathbb{E}_{F_1}(X) \geq \mathbb{E}_{F_2}(X).$$

Second-Order Stochastic Dominance

By choosing $u''(\cdot) = 0$, we see that $\mathbb{E}_{F_1}(X) \geq \mathbb{E}_{F_2}(X)$ is necessary.
If

$$\int_{\underline{x}}^{x'} F_1(y)\, dy > \int_{\underline{x}}^{x'} F_2(y)\, dy \text{ for some } x',$$

we have

$$\int_{\underline{x}}^{\overline{x}} u(x)[dF_1(x) - dF_2(x)] < 0,$$

if $u''(x') < 0$ and $u''(x) = 0$, for $x \notin (x', x' + \varepsilon)$, and $u'(\overline{x}) = 0$.
QED.

Second-Order Stochastic Dominance

If we allow all concave functions $u(\cdot)$ and consider

$$\Omega^{2'} = \left\{ u\left(x\right) : u'\left(x\right) \geq 0, u''\left(x\right) \leq 0 \right\},$$

we have

$$\int u\left(x\right) dF_1\left(x\right) \geq \int u\left(x\right) dF_2\left(x\right) \text{ for all } u\left(x\right) \in \Omega^{2'}$$

if and only if

$$\int_{\underline{x}}^{x} F_1\left(y\right) dx \leq \int_{\underline{x}}^{x} F_2\left(y\right) dy \text{ for all } y.$$

and

$$\mathbb{E}_{F_1}(X) = \mathbb{E}_{F_2}(X).$$

Second-Order Stochastic Dominance

For the case $u(\cdot) \in \Omega^{2'}$, we have:

Proposition Random variable $\widetilde{x}$ dominates random variable $\widetilde{y}$ for $u(\cdot) \in \Omega^{2'}$ if and only if we can write

$$\widetilde{y} = \widetilde{x} + \widetilde{z},$$

where $E\left[\widetilde{z}\,|\widetilde{x}\right] = 0$.

Somehow these characterizations for SOSD are associated with Rothschild and Stiglitz even though the first characterization is already in Hardy, Littlewood and Polya (1929) and the second is present in Blackwell's theorem.

An aside: Value of information

- Two states of nature: $\{\omega_0, \omega_1\}$. Let $p = \Pr\{\omega = \omega_0\}$
- A decision maker must choose $a \in A$
- Bernoulli utility function $u(\omega, a)$
- Expected utility from choosing a

$$U(a) = pu(\omega_0, a) + (1 - p)u(\omega_1, a).$$

- Value function:

$$V(p) = \max_{a \in A} pu(\omega_0, a) + (1 - p)u(\omega_1, a).$$

- $V(p)$ is convex.
- An experiment on Ω results in a (posterior) distribution p.
- Can you give a comparison of experiments along the lines of SOSD?

Applications: Standard Portfolio Choice

- Risk averse decision maker. Initial wealth w_0. Decision problem: Choose fraction α to invest in safe versus risky assets?
- No short sales allowed.
- $(1 + r)$ riskless return
- $(1 + \widetilde{r})$ the random return on the risky investment.
- Denote the amount of risky investment by $0 \leq \alpha \leq w_0$, and thus the safe investment is $(w_0 - \alpha)$.
- Final wealth of the decision maker:

$$(w_0 - \alpha)(1 + r) + a(1 + \widetilde{r}) = w_0(1 + r) + \alpha(\widetilde{r} - r).$$

Standard Portfolio Choice

- Strictly concave, strictly increasing twice differentiable Bernoulli utility function $u(w)$.
- Expected utility from a risky investment α:

$$v(\alpha) = Eu\left(w_0(1+r) + \alpha(\widetilde{r} - r)\right).$$

- $v(\alpha)$ is a strictly concave function of α if $\Pr(\widetilde{r} = r) < 1$ since

$$v''(\alpha) = E\left((\widetilde{r} - r)^2 u''\left(w_0(1+r) + \alpha(\widetilde{r} - r)\right)\right).$$

- FOC is thus also sufficient for maximum.

Standard Portfolio Choice

- The first order condition for interior solutions (i.e. for solutions where $0 < \alpha < w_0$):

$$v'(\alpha) = E(\widetilde{r} - r)\, u'(w_0(1+r) + \alpha(\widetilde{r} - r)) = 0.$$

- For $\alpha = 0$, it must be that

$$v'(0) = E(\widetilde{r} - r)\, u'(w_0(1+r)) \leq 0.$$

- Since $u'(w_0(1+r))$ is independent of $\widetilde{r}$, the above condition is equivalent to

$$u'(w_0(1+r))\, E(\widetilde{r} - r) \leq 0.$$

- Hence a necessary and sufficient condition for no risky investments is that the expected value of the investment be no larger than the safe return.
- Thus all decision makers, risk averse or not, invest some positive amount in risky assets if their expected return is larger than the safe rate.

Standard Portfolio Choice: The Effect of Risk Aversion

- Consider next two risk averse decision makers, $u_1$and u_2.
- Suppose that u_1 is more risk averse than u_2.
- Then $u_1(x) = \phi(u_2(x))$ for some concave function ϕ.
- We want to see how the optimal portfolio choices of u_1 and u_2 can be compared.
- Denote the optimal risky investments by α_1 and α_2 respectively.
- From the first order condition for u_2, we have:

$$v_2'(\alpha_2) = \mathbb{E}(\widetilde{r} - r)\, u_2'(w_0(1 + r) + \alpha_2(\widetilde{r} - r)) = 0. \quad (2)$$

Standard Portfolio Choice: The Effect of Risk Aversion

- To see how the optimal risky investment of u_1 relates to α_2, we evaluate the derivative of $v_1(\cdot)$ at $\alpha = \alpha_2$.

$$v_1'(\alpha_2) = \frac{d}{d\alpha}\mathbb{E}\phi\left(u_2\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)\right)$$

$$= \mathbb{E}\left(\tilde{r}-r\right)\phi'\left(u_2\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)\right)$$

$$\cdot u_2'\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right).$$

- Since $\phi'' \leq 0$, we know that

$$\left(\tilde{r}-r\right)\phi'\left(u_2\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)\right) \leq \left(\tilde{r}-r\right)\phi'\left(u_2\left(w_0\left(1+r\right)\right.\right.$$

for all $\tilde{r}$.

Standard Portfolio Choice: The Effect of Risk Aversion

- To see this note that for $\tilde{r} < r$,

$$\phi'\left(u_2\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)\right) \geq \phi'\left(u_2\left(w_0\left(1+r\right)\right)\right)$$

  by the concavity of ϕ and similarly for $\tilde{r} > r$,

$$\phi'\left(u_2\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)\right) \leq \phi'\left(u_2\left(w_0\left(1+r\right)\right)\right)$$

  and hence the claim follows.

- But then we know that

$$v_1'\left(\alpha_2\right) \leq \mathbb{E}\left(\tilde{r}-r\right)\phi'\left(u_2\left(w_0\left(1+r\right)\right)\right)u_2'\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right)$$

$$= \phi'\left(u_2\left(w_0\left(1+r\right)\right)\right)\mathbb{E}\left(\tilde{r}-r\right)u_2'\left(w_0\left(1+r\right)+\alpha_2\left(\tilde{r}-r\right)\right) = 0,$$

  where the last equality follows from 2. Thus by the concavity of $v_1\left(\alpha\right)$, we know that $\alpha_1 \leq \alpha_2$.

Standard Portfolio Choice: The Effect of Risk Aversion

Proposition If u_1 is more risk averse than u_2, then $\alpha_1 \leq \alpha_2$ in the standard portfolio problem.

This proposition also yields an immediate corollary for risky investment as a function of initial wealth.

Corollary If u satisfies decreasing absolute risk aversion, then $\alpha(w_0) \leq \alpha(w_0')$ whenever $w_0 < w_0'$.

proof Take $u_2(z) = u(z)$ and $u_1(z) = u(z - k)$ and apply the previous theorem.

Applications: Consumption and Savings

- Deterministic two-period model with additively separable utility function.
- Separate Bernoulli utility function for the consumption in each period $t = 0, 1$.
- Consumer receives wealth w_0 and w_1 respectively in the two periods.
- She can borrow and lend as she wishes at the risk free rate r.
- If we let s denote the savings by the consumer, then the optimization problem can be written as

$$\max_{s} \; u_0\left(w_0 - s\right) + u_1\left(w_1 + s\left(1 + r\right)\right).$$

Consumption and Savings

- Observe that we can allow for negative saving (i.e. borrowing) in this model, but we require that consumption be positive in both periods (i.e. $s \leq w_0$).
- Assume throughout that $u_i(\cdot)$ are strictly concave and twice continuously differentiable for $i = 0, 1$.
- Hence if we let

$$v(s) = u_0(w_0 - s) + u_1(w_1 + s(1 + r)),$$

  we see immediately that $v''(s) < 0$.
- This allows us again to locate optimal savings levels from the first order conditions.
- The optimal level of savings s^* is characterized by

$$v'(s^*) = -u_0'(w_0 - s^*) + (1 + r)\, u_1'(w_1 + s^*(1 + r)) = 0.$$

Consumption and Savings

- If $u_0 = u_1 = u$ and $r = 0$, we see the most clearly how savings are used to smooth consumption across periods. From
$$u'(w_0 - s^*) = u'(w_1 + s^*),$$
we conclude by the strict concavity of u that
$$w_0 - s^* = w_1 + s^*.$$
- Hence the consumption levels in the two periods are identical.
- The other main motive of saving is to increase wealth.
- This effect can obviously only be seen when $r > 0$.
- Again in the case where $u_0 = u_1 = u$, we get
$$u'(w_0 - s^*) = (1 + r)\, u'(w_1 + s^*(1 + r)).$$
- By the concavity of u, we see that consumption in the second period is larger (since the marginal utility is lower) than in the first period.

Consumption and Savings

- Hence the consumer is willing to sacrifice some of the consumption smoothing for increases in wealth.
- Finally, we can totally differentiate the first order condition with respect to s and w_i to get

$$\frac{ds^*}{dw_0} = \frac{u_0''\left(w_0 - s^*\right)}{\left[u_0''\left(w_0 - s^*\right) + \left(1+r\right)^2 u_1''\left(w_1 + s^*\left(1+r\right)\right)\right]} > 0,$$

$$\frac{ds^*}{dw_1} = \frac{-u_1''\left(w_1 + s^*\left(1+r\right)\right)}{\left[u_0''\left(w_0 - s^*\right) + \left(1+r\right)^2 u_1''\left(w_1 + s^*\left(1+r\right)\right)\right]} < 0.$$

- Hence an increase in the first period income increases savings, and an increase in the second period income decreases savings.

Consumption and Savings

- With these preliminaries in place, we can start the analysis of the optimal savings problem in a world of uncertainty.
- The first question that we ask is whether the optimal savings are larger in a model where the second period income is random than in the deterministic model.

Definition A utility function is prudent if adding an uninsurable zero mean risk to the second period income increases the savings.

- To characterize prudent utility functions, let $\widetilde{w}_1 = w_1 + \widetilde{r}$, where $\widetilde{r}$ is assumed to be uninsurable and $E\widetilde{r} = 0$.
- Denote the new expected utility from savings s by:

$$V(s) = u_0(w_0 - s) + Eu_1(w_1 + s(1 + r) + \widetilde{r}).$$

- $V(s)$ inherits the curvature of the u_i functions.

Consumption and Savings

- Analyze comparative static questions by evaluating the derivative of $V(s)$ at point s^* such that $v'(s^*) = 0$, i.e. at the optimal savings level of the deterministic model.
- Observe that $V'(s^*) \geq 0$ if

$$Eu_1'\left(w_1 + s^*\left(1+r\right) + \widetilde{r}\right) \geq u_1'\left(w_1 + s^*\left(1+r\right)\right). \quad (3)$$

- Notice that on the left hand side of the inequality, we have the expected utility from a random variable.
- On the right hand side, we have the utility from the expected value of the random variable.
- This is exactly the definition of a risk loving utility function since w_1 and $\widetilde{r}$ are arbitrary.
- As risk loving functions are convex, we deduce that 3 holds for all w_1 and $\widetilde{r}$ if and only if u_1' is convex.

Consumption and Savings

Hence we have proved the following proposition.

Proposition A utility function is prudent if and only if u_1' is convex.

- From this point on, we could develop a theory for comparing prudence of different individuals or the prudence of a given individual at various wealth levels.
- Much of this theory has been done by Miles Kimball, and the central concept for the analysis is the coefficient of absolute prudence:

$$P(w) = \frac{-u'''(w)}{u''(w)}.$$

- We conclude this section on precautionary savings by recalling from the previous lecture the derivation for decreasing absolute risk aversion.

$$\frac{d}{dw} r^A(w) = r^A(w)\left[r^A(w) - P(w)\right].$$

Consumption and Savings

- Hence there are two arguments for believing in the prevalence of prudent utility functions.
- First of all, there is direct econometric evidence on the savings behavior of individuals with various degrees of uninsurable risk positions.
- Second, there is overwhelming empirical support for decreasing absolute risk aversion.
- As the formula above indicates, DARA is only possible for prudent utility functions.