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**Handout on Stochastic Dominance**

Since the textbook is less than stellar in its treatment of the notions of stochastic dominance, it may be a good idea to collect some of the basic ideas in a handout. These notes owe a lot to the treatment of this topic in 'The Economics of Risk and Time' by Christian Gollier, MITPress, 2001.

The point of the exercise is to obtain a classification of risks in such a way that all utility functions in a prespecified class of functions rank the risks in the same manner. The notions of stochastic dominance will thus be different depending on the class of utility functions that are considered. The two most commonly used notions are first order stochastic dominance and second order stochastic dominance. These are the two notions that were handled in class as well.

Hence if $\Omega$ is the set of utility functions that are of interest, we are after characterizations for distributions $dF_1(x)$ and $dF_2(x)$ to guarantee that

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ for all } u(x) \in \Omega. \tag{1}$$

Since the set $\Omega$ may be a large set, it would be helpful if we could obtain the desired conclusion based on evaluating the integrals in 1 for a smaller set of functions. The key to finding such smaller sets is to note that expected utilities are linear in the utility functions. In other words, if

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ and}$$

$$\int \widehat{u}(x)\,dF_1(x) \geq \int \widehat{u}(x)\,dF_2(x), \text{ then}$$

$$\int [\alpha u(x) + \beta \widehat{u}(x)]\,dF_1(x) \geq \int [\alpha u(x) + \beta \widehat{u}(x)]\,dF_2(x).$$

Thus we want to find a basis for $\Omega$, i.e. a set of functions $\{b(x,\theta)\}_{\theta\in\Theta}$, such that all $u \in \Omega$ can be expressed as linear combinations of functions in $\{b(x,\theta)\}_{\theta\in\Theta}$. If $\Theta$ is uncountable, then the appropriate way of taking linear combinations of functions in $\{b(x,\theta)\}_{\theta\in\Theta}$ is by taking integrals over $\Theta$. Hence we want to find a set of functions $\{b(x,\theta)\}_{\theta\in\Theta}$ such that for all $u \in \Omega^1$, we have:

$$u(x) = \int b(x,\theta)\, dH(\theta)$$

for some "distribution" $dH(\theta)$.[1] If we know that

$$\int b(x,\theta)\, dF_1(x) \geq \int b(x,\theta)\, dF_2(x) \text{ for all } \theta,$$

then we know immediately that[2]

$$\int u(x)\, dF_1(x) = \int \int b(x,\theta)\, dH(\theta)\, dF_1(x)$$

$$= \int \int b(x,\theta)\, dF_1(x)\, dH(\theta) \geq \int \int b(x,\theta)\, dF_2(x)\, dH(\theta)$$

$$= \int \int b(x,\theta)\, dH(\theta)\, dF_2(x) = \int u(x)\, dF_2(x).$$

In the remainder of this handout, I will specialize this approach to two classes of Bernoulli utility functions.

**First order stochastic dominance**

For this notion, the utility functions of interest are all increasing functions, i.e. we consider

$$\Omega^1 = \left\{u(x) : u'(x) \geq 0\right\}.$$

We will assume that $x \in [\underline{x}, \overline{x}]$ and that $u$ is differentiable throughout.[3]

[1] It is not really a distribution as it does not have to integrate to unity.

[2] The second equality is Fubini's theorem on interchanging the order of integration.

[3] The notions generalize to unbounded supports and nondifferentiable functions as long as the integrals in 1 converge.

**Definition 1** *A distribution $dF_1(x)$ first order stochastically dominates $dF_2(x)$ if*

$$\int u(x)\,dF_1(x) \geq \int u(x)\,dF_2(x) \text{ for all } u(x) \in \Omega^1.$$

For this set of Bernoulli utility functions, I claim that a basis is given by indicator functions

$$b(x,\theta) = I(x \geq \theta),$$

$$\text{where } I(x \geq \theta) = \begin{cases} 1 \text{ if } x \geq \theta, \\ 0 \text{ if } x \leq \theta. \end{cases}$$

The reasoning for thinking that this might be a suitable basis for increasing functions goes as follows. The Indicator functions are constant everywhere except at a single point $\theta$. At the point of discontinuity, the indicator function jumps up by a unit. Hence it might be reasonable to think that by pasting together these jumps, we can recover any increasing function. To show that this indeed can be done, I construct the weight function $dH(\theta)$ for an arbitrary function $u \in \Omega^1$.

To do this, recall from the fundamental theorem of differential and integral calculus that

$$u(x) = u(\underline{x}) + \int_{\underline{x}}^{x} u'(\theta)\,d\theta.$$

There is no loss of generality in normalizing all the utility functions to have $u(\underline{x}) = 0$ (exercise: why?).Hence

$$u(x) = \int_{\underline{x}}^{x} u'(\theta)\,d\theta = \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\,u'(\theta)\,d\theta.$$

Hence the weight function $dH(\theta)$ is given by $u'(\theta)$.[4]

With this basis, we can easily characterize when $dF_1(x)$ first order stochastically dominates $dF_2(x)$. We must have

$$\int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\,dF_1(x) \geq \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\,dF_2(x) \text{ for all } \theta.$$

[4]If we didn't assume differentiability, the weight function would have a mass point at any discontinuity $x$ of size $u(x+) - u(x-)$ where the expressions are for right and left limits at the discontinuity. (They exist since $u$ is assumed to be increasing).

In other words,

$$\int_{\theta}^{\overline{x}} dF_1(x) \geq \int_{\theta}^{\overline{x}} dF_2(x) \text{ for all } \theta, \text{ or}$$

$$1 - F_1(\theta) \geq 1 - F_2(\theta) \text{ for all } \theta, \text{ or}$$

$$F_1(\theta) \leq F_2(\theta) \text{ for all } \theta.$$

Thus we have proved the following proposition.

**Proposition 2** *$dF_1(x)$ first order stochastically dominates $dF_2(x)$ if and only if*

$$F_1(x) \leq F_2(x) \text{ for all } x \in [\underline{x}, \overline{x}].$$

**Second order stochastic dominance**

For this notion, the relevant class of Bernoulli utility functions is given by $\Omega^2 = \{u(x) : u'(x) \geq 0, u''(x) \leq 0\}$.

**Definition 3** *A distribution $dF_1(x)$ second order stochastically dominates $dF_2(x)$ if*

$$\int u(x)\, dF_1(x) \geq \int u(x)\, dF_2(x) \text{ for all } u(x) \in \Omega^2.$$

For this set of functions, I claim that functions $b(x, \theta) = \min(x, \theta)$ form a basis. The reason for this choice for a basis is the following. Increasing concave functions have decreasing first derivatives. We can then apply the reasoning of the previous section to the first derivatives (the derivatives of $\min(x, \theta)$ functions are indicator functions with a jump down at $\theta$). Again, I construct the weight function for an arbitrary function in $\Omega^2$.

Consider the expression

$$\int_{\underline{x}}^{\overline{x}} \min(x, \theta)\, dH(\theta). \tag{2}$$

Integration by parts yields:

$$\int_{\underline{x}}^{\overline{x}} \min(x, \theta)\, dH(\theta) = |_{\underline{x}}^{\overline{x}} \min(x, \theta)\, H(\theta) - \int_{\underline{x}}^{\overline{x}} I(x \geq \theta)\, H(\theta)\, d\theta$$

$$= xH(\overline{x}) - \underline{x}H(\underline{x}) - \int_{\underline{x}}^{x} H(\theta)\, d\theta.$$

Thus if I choose

$$u'(x) = H(\overline{x}) - H(x),$$

$$-u''(x) = dH(x),$$

I have the appropriate weight function.[5,6] (It is a good exercise for the interested reader to check that this really works. To do this, try an arbitrary concave function, calculate its second derivative, plug into 2. After the appropriate integrations, you should recover the original utility function).

The remaining step is to derive the implications from

$$\int_{\underline{x}}^{\overline{x}} \min(x,\theta)\, dF_1(x) \geq \int_{\underline{x}}^{\overline{x}} \min(x,\theta)\, dF_2(x) \text{ for all } \theta. \tag{3}$$

Integrating the left hand side by parts yields:

$$|_{\underline{x}}^{\overline{x}} \min(x,\theta)\, F_1(x) - \int_{\underline{x}}^{\overline{x}} I(x \leq \theta)\, F_1(x)\, dx$$

$$= \theta - \int_{\underline{x}}^{\theta} F_1(x)\, dx.$$

Similarly for the right hand side, we get:

$$\theta - \int_{\underline{x}}^{\theta} F_2(x)\, dx.$$

Hence we get that 3 hold if and only if

$$\theta - \int_{\underline{x}}^{\theta} F_1(x)\, dx \geq \theta - \int_{\underline{x}}^{\theta} F_2(x)\, dx \text{ for all } \theta \text{ or}$$

$$\int_{\underline{x}}^{\theta} F_1(x)\, dx \leq \int_{\underline{x}}^{\theta} F_2(x)\, dx \text{ for all } \theta.$$

We have thus proved the following proposition.

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[5] Observe from this formula, that we will have $u'(\underline{x}) = H(\overline{x})$, and $u'(\overline{x}) = H(b) - \lim_{\theta \to \overline{x}-} H(\theta)$. Hence if $u'(\overline{x}) > 0$, $H(\theta)$ will have a mass point of sixe $u'(\overline{x})$ at $\theta = \overline{x}$.

[6] At any point $x$ where $u(x)$ is not differentiable, $H(\theta)$ has a mass point of size $\lim u'(x+) - \lim u'(x-)$. These limits exist for all $x$ since $u(x)$ is a concave function.

**Proposition 4** *$dF_1(x)$ second order stochastically dominates $dF_2(x)$ if and only if*

$$\int_{\underline{x}}^{\theta} F_1(x)\,dx \leq \int_{\underline{x}}^{\theta} F_2(x)\,dx \text{ for all } \theta.$$

To describe another characterization of second order stochastic dominance, it will be useful to give the conditions in terms of random variables rather than their distributions. We say that a random variable second order dominates another random variable if the distribution of the first second order stochastically dominates the distribution of the second.

**Proposition 5** *Random variable $\widetilde{x}$ second order stochastically dominates random variable $\widetilde{y}$ if and only if we can write*

$$\widetilde{y} = \widetilde{x} + \widetilde{z},$$

*where $E[\widetilde{z}\,|\widetilde{x}] = 0$.*

It is easy to see sufficiency (i.e. that if $\widetilde{y} = \widetilde{x} + \widetilde{z}$, then $\widetilde{x}$ second order stochastically dominates $\widetilde{y}$). This follows immediately from Jensen's inequality. Rothschild and Stiglitz: 'Increasing Risk: I. A Definition', JET, 1970 show necessity of the condition as well.

Finally, it should be noted that a similar classification of risks for convex utility functions is possible. Then all the inequalities in the treatment are reversed.

### An illustration of the techniques

Consider a maximization problem where the utility depends on the realization of a random variable as well as the chosen value of a control variable. Let $u(a,x)$ be the utility function. In many applications, the following assumptions are natural to make: $u_a(a,x) > 0$ and $u_{aa}(a,x) < 0$. Let $a$ denote the control, and let $F(x,r)$ be the distribution of the random variable with the risk parameter $r$. Assume that higher values of $r$ represent risks that are second order stochastically dominated by risks with lower $r$.The problem is then to

$$\max_a \int_0^1 u(a,x)\,dF(x,r) = \max_a \int_0^1 u(a,x)\,f(x,r)\,dx,$$

where $f(x,r)$ is the density function of the random variable. The first derivative of the expected utility from choice $a$ is given by

$$v'(a) = \int_0^1 u_a(a,x) f(x,r)\, dx.$$

Observe that

$$v''(a) < 0 \text{ since } u_{aa}(a,x) < 0.$$

If we are interested in the comparative statics of the control variable in the risk parameter, we need to apply the results that we had for second order stochastic dominance above. Define $a(r)$ to be the optimal choice for risk level $r$. Then

$$v'(a(r)) = \int_0^1 u_a(a,x) f(x,r)\, dx = 0.$$

We know that if $u_a(a,x)$ is concave in $u$, then

$$\int_0^1 u_a(a(r),x) f(x,r')\, dx \leq 0 \text{ for } r' \geq r.$$

This simple reflects the fact that the expected utility for concave functions is lower from risks that are second order stochastically dominated (i.e. the proposition of the previous section). Since $v''(a)$ is concave, this implies that $a(r') < a(r)$. Hence the optimal action is decreasing in $r$. The opposite conclusion follows if $u_a(a,x)$ is convex in $x$. To summarize, we have shown the following proposition.

**Proposition 6** *Let the utility from action $a$ and outcome $x$ is given by a function $u(a,x)$ and the distribution of the risk is given by $F(x,r)$, where $r$ is a parameter of increasing risk in the sense of second order stochastic dominance. The optimal choice $a(r)$ as a function of the risk is*

*i) decreasing if $u_{axx}(a,x) \leq 0$,*

*ii) increasing if $u_{axx}(a,x) \geq 0$.*