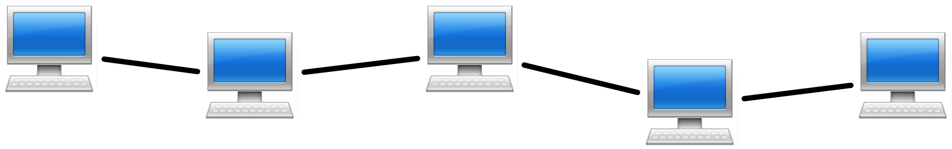
- **Weeks 1–2: informal introduction**
  - network = path 
- **Week 3: graph theory**
- **Weeks 4–7: models of computing**
  - what can be computed (efficiently)?
- **Weeks 8–11: lower bounds**
  - what cannot be computed (efficiently)?
- **Week 12: recap**

Week 5

- LOCAL model:
  unique identifiers

LOCAL model

- Idea: nodes have unique names
- Names arbitrary but fairly short
- IPv4 addresses, IPv6 addresses, MAC addresses, IMEI numbers…

LOCAL model

- **LOCAL model =**
  PN model + unique identifiers
- **Assumption: unique identifiers**
  are given as local inputs

LOCAL model

- **Algorithm has to work correctly for any port numbering and for any unique identifiers**
- **Adversarial setting:**
  - you design algorithms
  - adversary picks graph, port numbering, IDs

LOCAL model

- Fixed constant c
- In a network with n nodes, identifiers are a subset of $\{1, 2, \ldots, n^c\}$
- Equivalently: unique identifiers can be encoded with $O(\log n)$ bits

PN vs. LOCAL

- **PN: few problems can be solved**
- **LOCAL: all problems can be solved**
  (on connected graphs)

PN vs. LOCAL

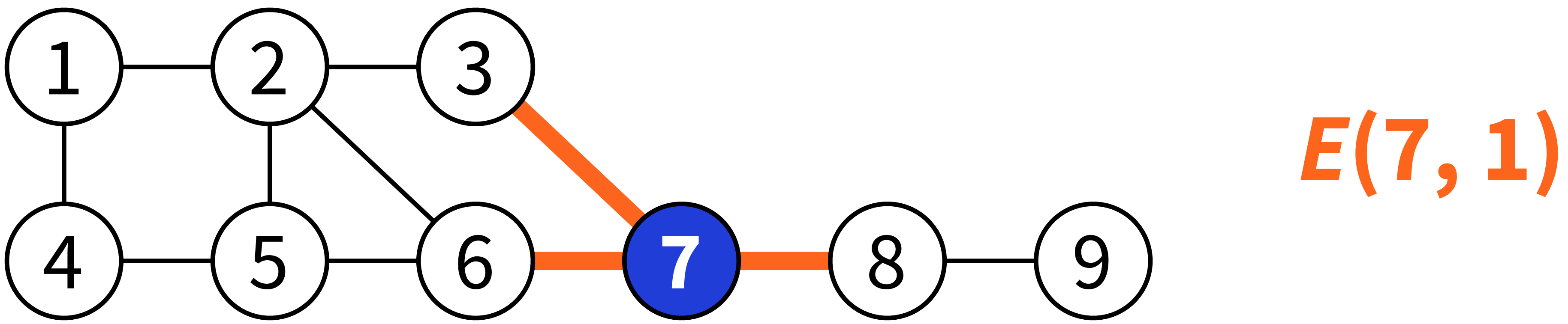
- **PN: "*what can be computed?*"**
- **LOCAL: "*what can be computed efficiently?*"**

Solving everything

- **All nodes learn everything about the graph**
  - $O(\mathrm{diam}(G))$ rounds
- **All nodes solve the problem locally (e.g., by brute force)**
  - 0 rounds

Gathering everything

- $E(v, r)$ = "edges within distance r from v"
  = one endpoint at distance at most $r - 1$ from v



Gathering everything

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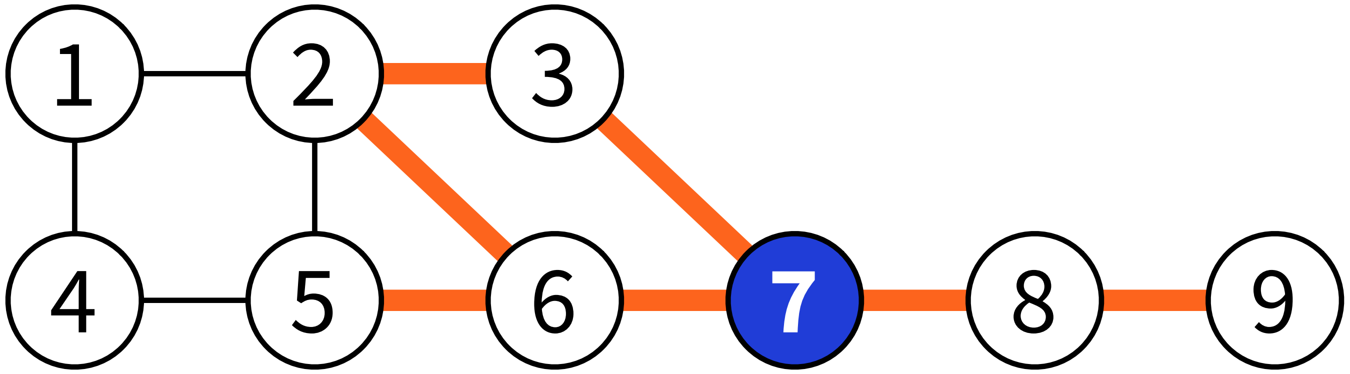

$E(7, 2)$

Gathering everything

- $E(v, r)$ = "edges within distance r from v"
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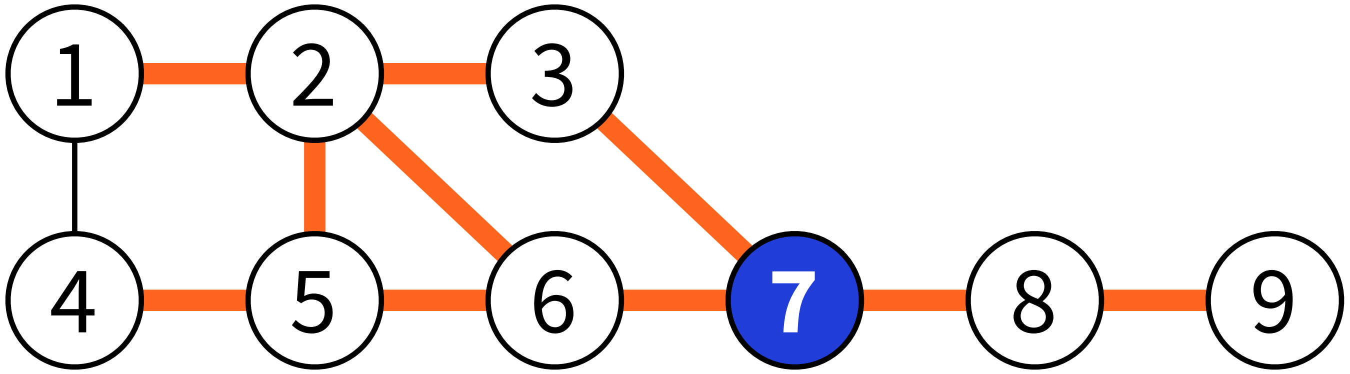

$E(7, 3)$

Gathering everything

- $E(v, r)$ = "edges within distance r from v"
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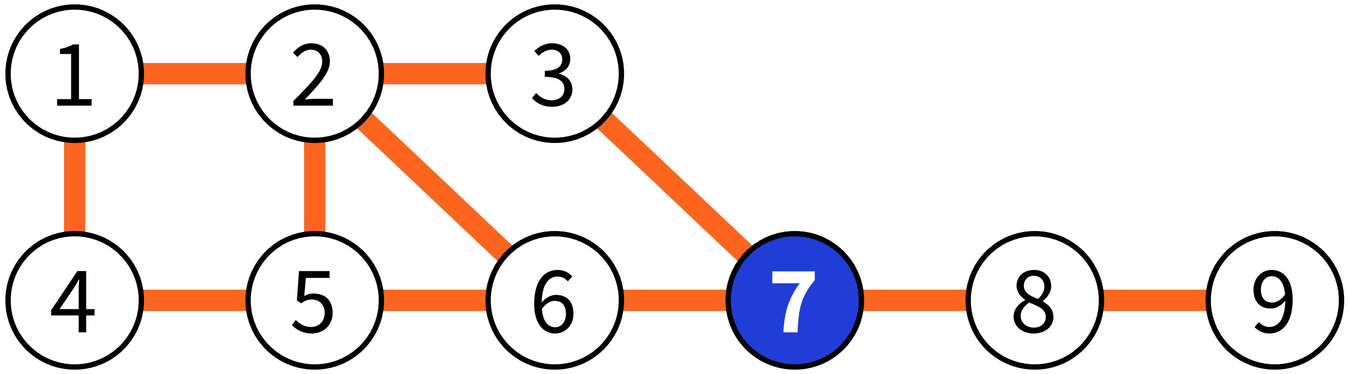
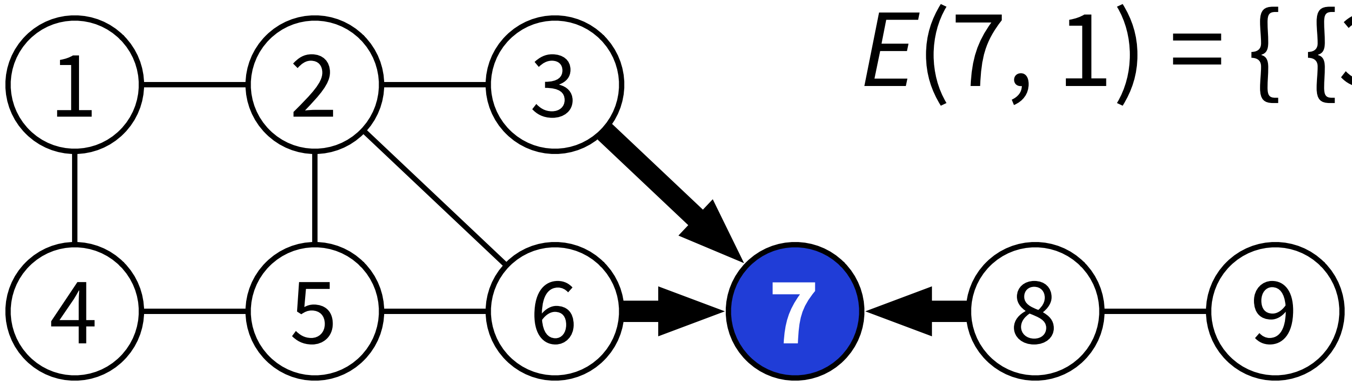


$E(7, 4)$

Gathering everything

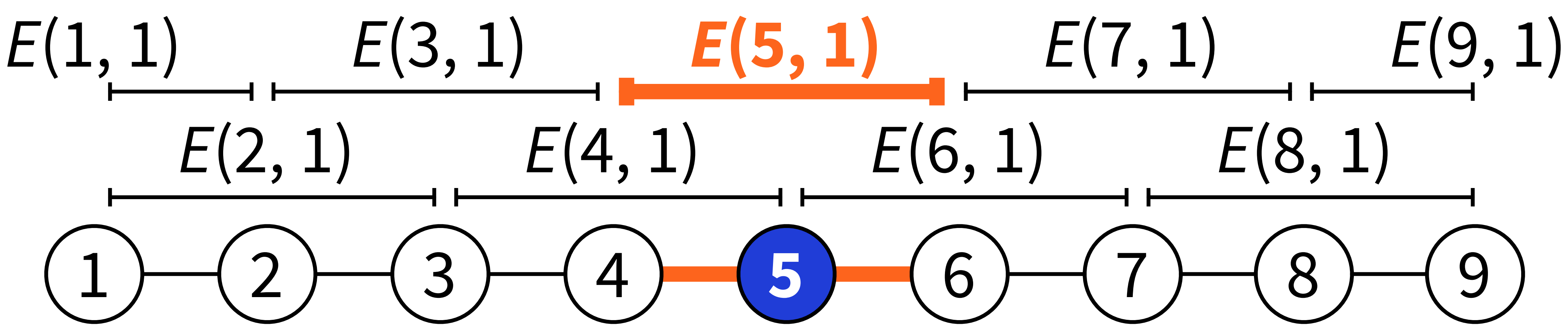
- **Each node *v* can learn *E*(*v*, 1) in 1 round**
  - send own ID to all neighbours



$E(7, 1) = \{ \{3, 7\}, \{6, 7\}, \{7, 8\} \}$

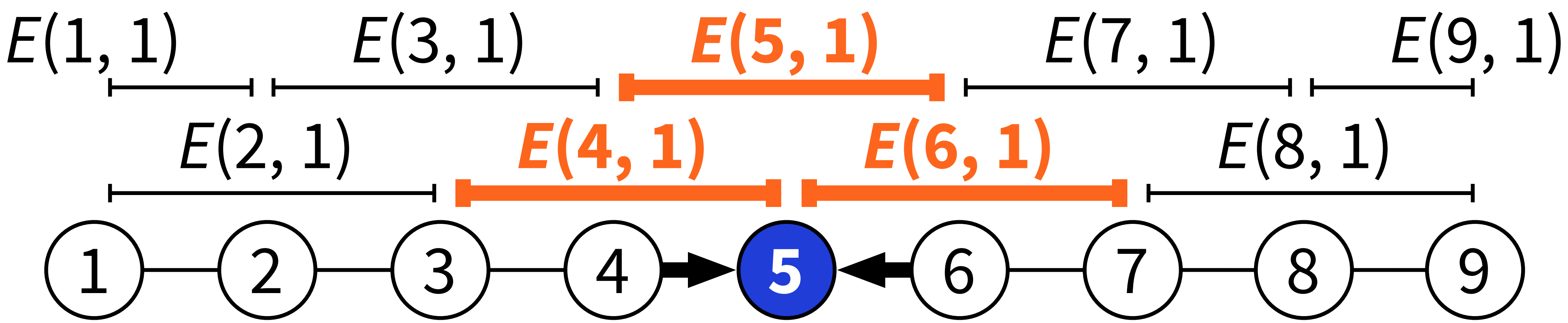
Gathering everything

- **Each node *v* can learn $E(v, 1)$ in 1 round**
  - send own ID to all neighbours



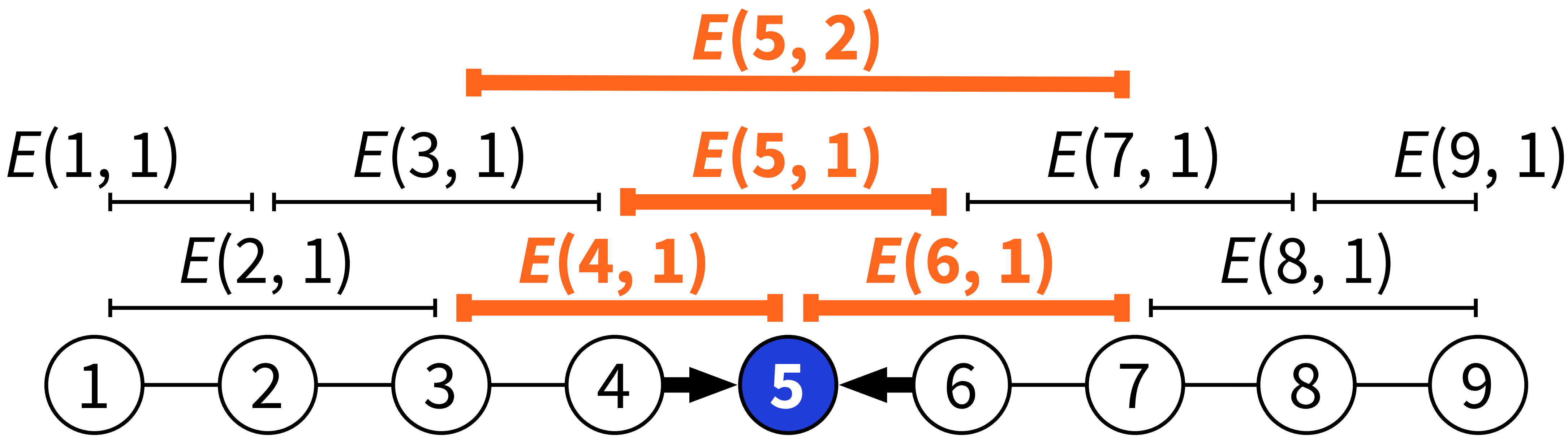
Gathering everything

- **Given $E(v, r)$, we can learn $E(v, r + 1)$ in 1 round**
  - send $E(v, r)$ to all neighbours, take union



Gathering everything

- **Given $E(v, r)$, we can learn $E(v, r + 1)$ in 1 round**
  - send $E(v, r)$ to all neighbours, take union



Gathering everything

- **One of the following holds:**
  - $E(v, r) \neq E(v, r+1)$: **learn something new**
  - $E(v, r) = E(v, r+1) = E$: **we can stop**
- **Proof idea:**
  - if $E(v, r) \neq E$, there are unseen edges adjacent to $E(v, r)$, and they will be in $E(v, r+1)$

Example: **Graph colouring**

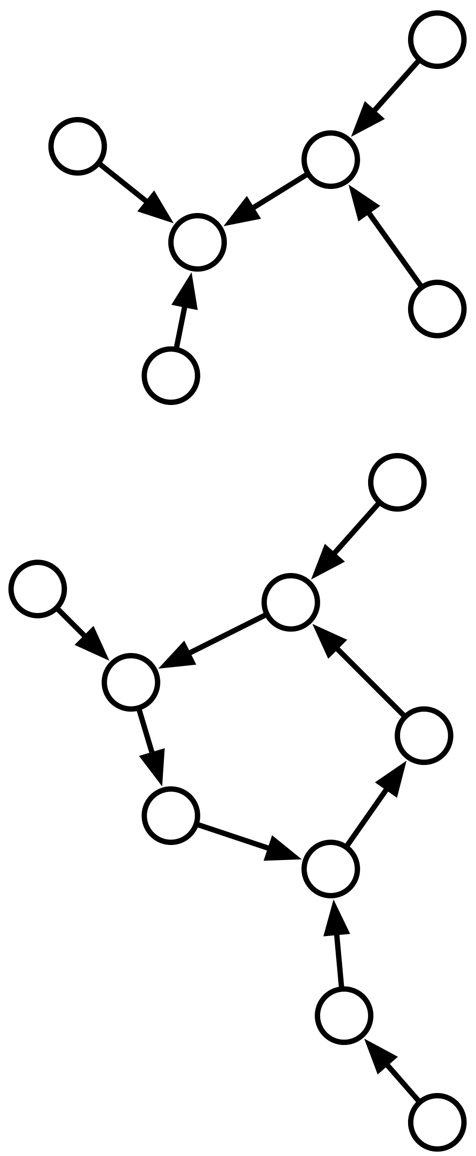
- **We can solve everything in $O(\text{diam}(G))$ time**
- **What can be solved much faster?**
- **Example: graph colouring with $\Delta + 1$ colours**
  - can be solved in $O(\Delta^{0.51} + \log^* n)$ rounds
  - today: how to do it in $O(\Delta^2 + \log^* n)$ rounds?

Example:
Graph colouring

- **Setting: LOCAL model, *n* nodes, any graph of maximum degree Δ**
- **We assume that *n* and Δ are known**
  - if not known: guess some *n* and Δ, colour what you can, increase *n* and Δ, …

Directed pseudoforest

- **Directed graph, outdegree ≤ 1**
- **Each node has at most one “successor”**
- **Easy to 3-colour in time $O(\log^* n)$, we will use this as subroutine**



Directed pseudoforest

- **Colouring directed pseudoforests almost as easy as colouring directed paths**

- **Recall path-colouring algorithm P3CBit…**

Algorithm P3CBit:
Fast colour reduction

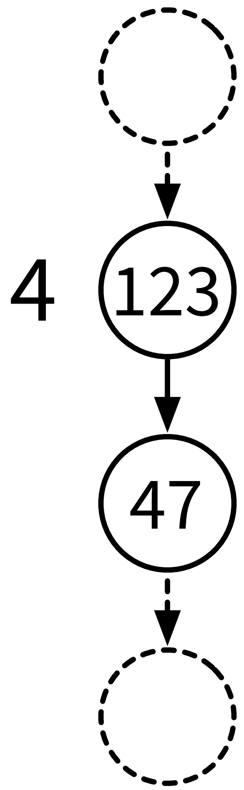
$\boldsymbol{c_0} = \mathbf{123} = \mathbf{01111011_2}$ (my colour)

$\boldsymbol{c_1} = \mathbf{47} = \mathbf{00101111_2}$ (successor's colour)

$\boldsymbol{i} = \mathbf{2}$ (bits numbered 0, 1, 2, … from right)

$\boldsymbol{b} = \mathbf{0}$ (in my colour bit number *i* was 0)

$\boldsymbol{c} = \mathbf{2 \cdot 2 + 0 = 4}$ (my new colour)



$k = 8$, reducing from $2^8 = 256$ to $2 \cdot 8 = 16$ colours

Directed pseudoforest

- **Colouring directed pseudoforests almost as easy as colouring directed paths**
- **Recall path-colouring algorithm P3CBit:**
  - nodes **only look at their successor**
  - my new colour ≠ successor's new colour
  - works equally well in directed pseudoforests!

Algorithm DPBit:
Fast colour reduction

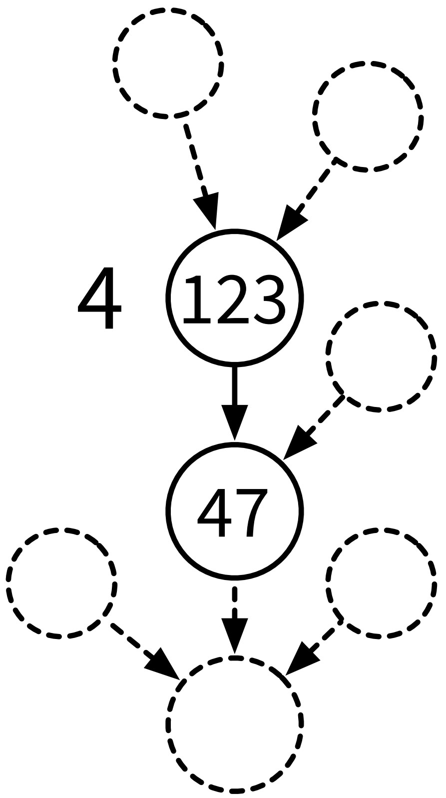
c_0 = **123** = **01111011**$_2$ (my colour)

c_1 = **47** = **00101111**$_2$ (successor's colour)

i = **2** (bits numbered 0, 1, 2, … from right)

b = **0** (in my colour bit number i was 0)

$c = 2 \cdot 2 + 0 = 4$ (my new colour)



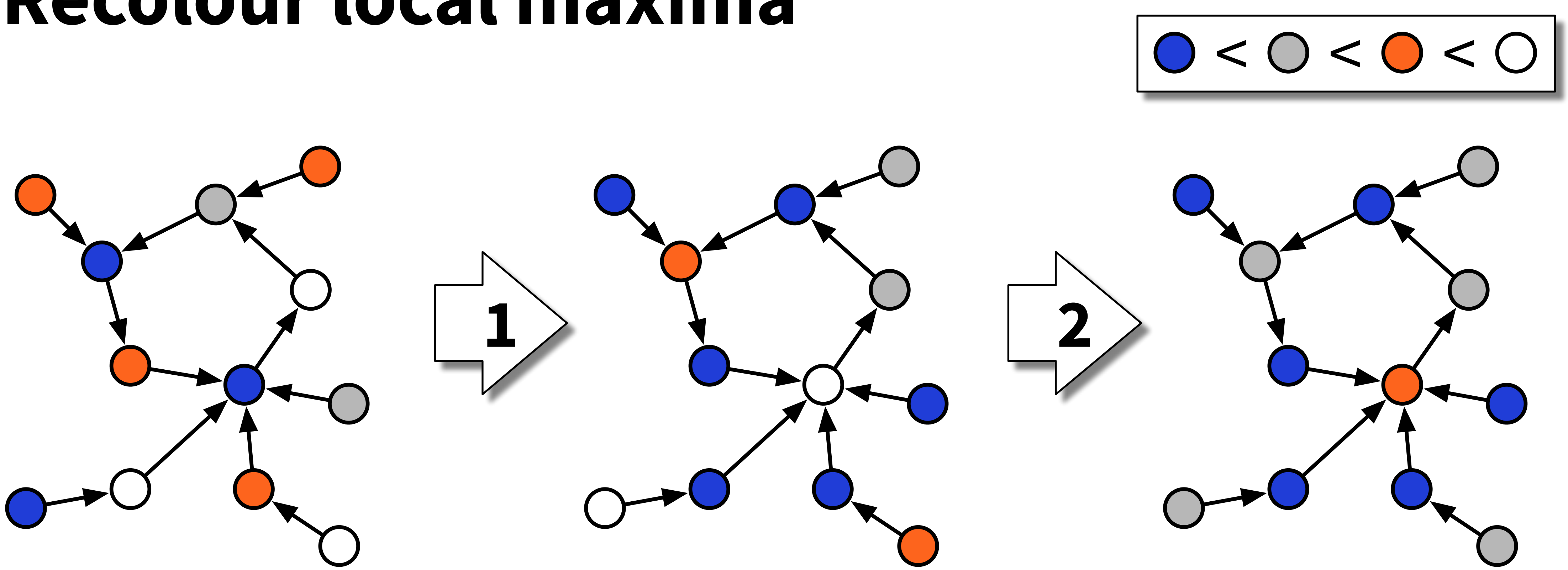
$k = 8$, reducing from $2^8 = 256$ to $2 \cdot 8 = 16$ colours

Directed pseudoforests

- Unique identifiers = $n^{O(1)}$ colours
- Iterate DPBit for $O(\log^* n)$ steps
  to reduce the number of colours to 6
- Apply a greedy algorithm
  to reduce the number of colours to 3

Algorithm DPGreedy: **Slow colour reduction**

1. **Shift:** ***predecessors have the same colour***
2. **Recolour local maxima**



Directed pseudoforests

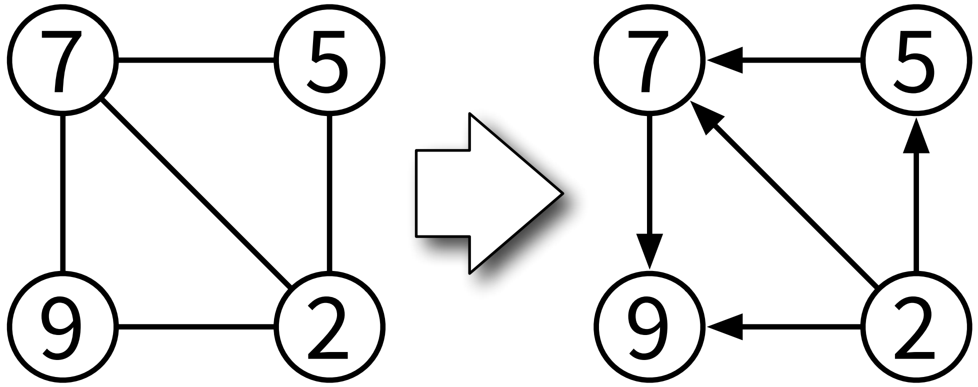
- Unique identifiers = $n^{O(1)}$ colours
- Iterate DPBit for $O(\log^* n)$ steps to reduce the number of colours to 6
- Iterate DPGreedy for 3 steps to reduce the number of colours to 3

Algorithm BDColour: **Fast graph colouring**

- **Unique identifiers → orientation**
- **Port numbers → partition edges in Δ directed pseudoforests**
- **3-colour pseudoforests in time $O(\log^* n)$**
- **Merge pseudoforests in time $O(\Delta^2)$**

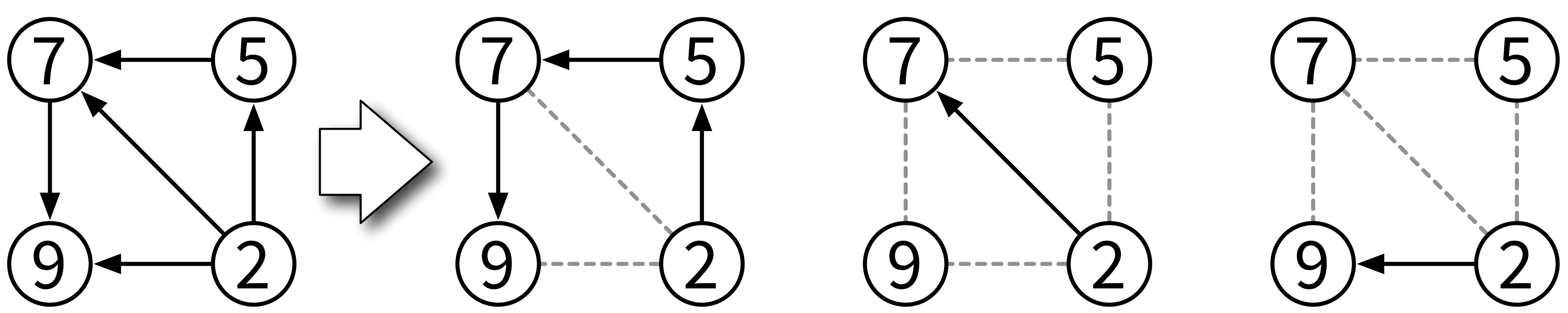
Algorithm BDColour: **Fast graph colouring**

- **Unique identifiers → orientation**
  - edges directed from smaller to larger ID



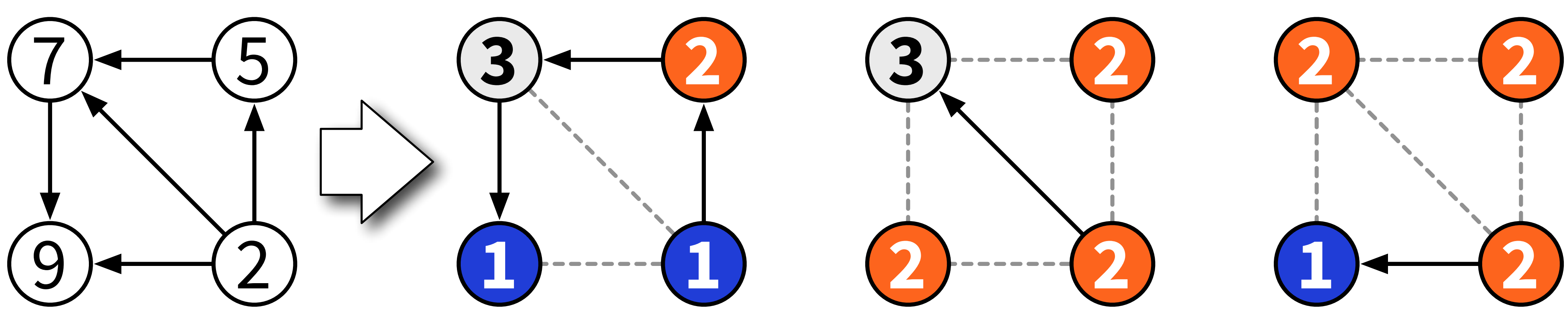
Algorithm BDColour: **Fast graph colouring**

- **Port numbers → partition edges in Δ directed pseudoforests**
  - *k*th outgoing edge → *k*th pseudoforest



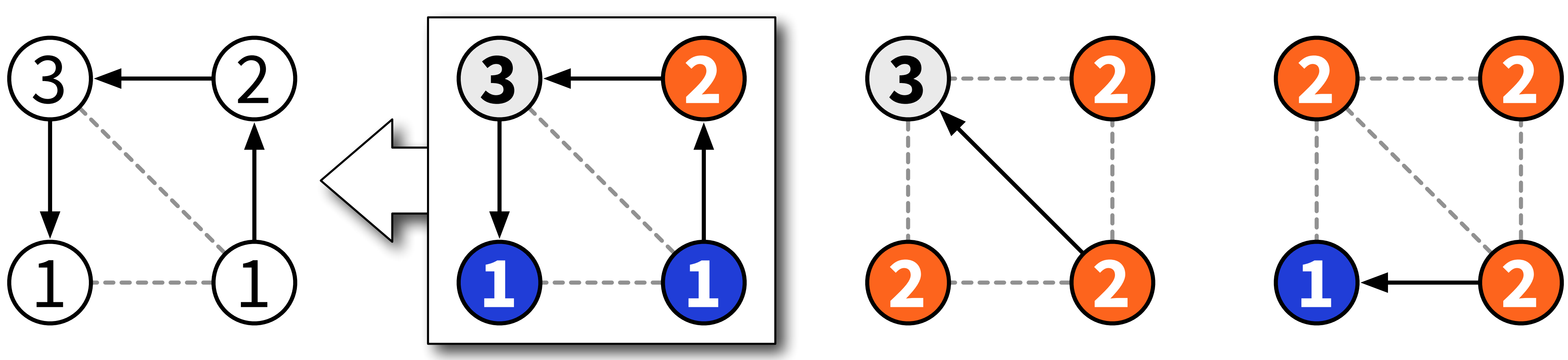
Algorithm BDColour: **Fast graph colouring**

- **3-colour pseudoforests in time $O(\log^* n)$**
  - all in parallel
  - each node has Δ roles



Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
  - add first forest: trivial



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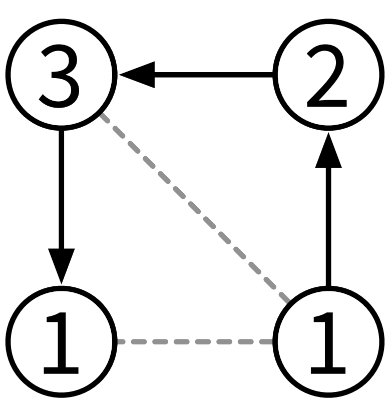

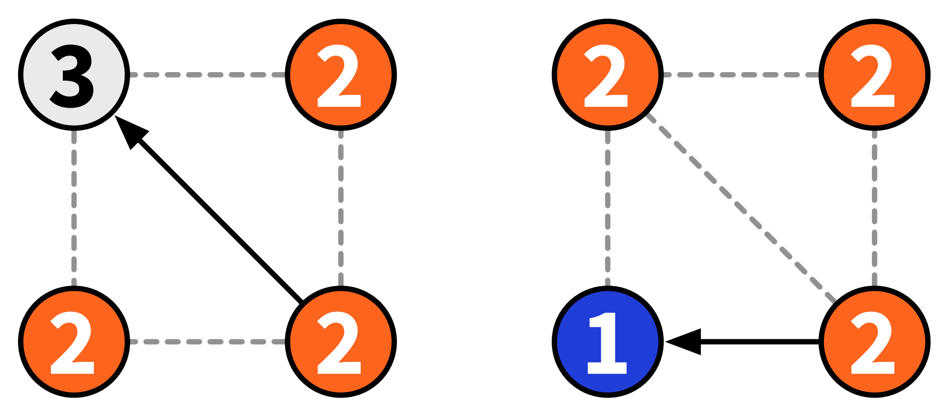

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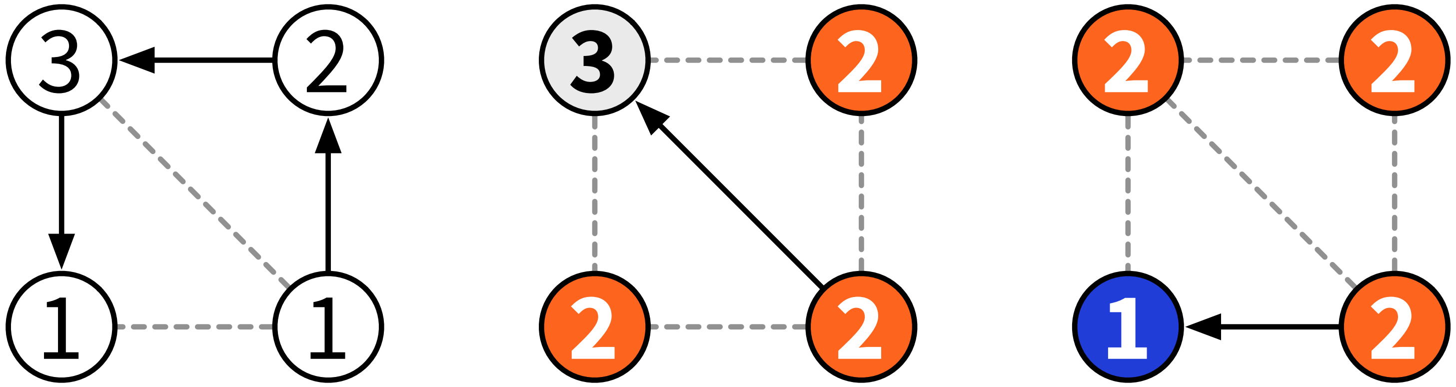
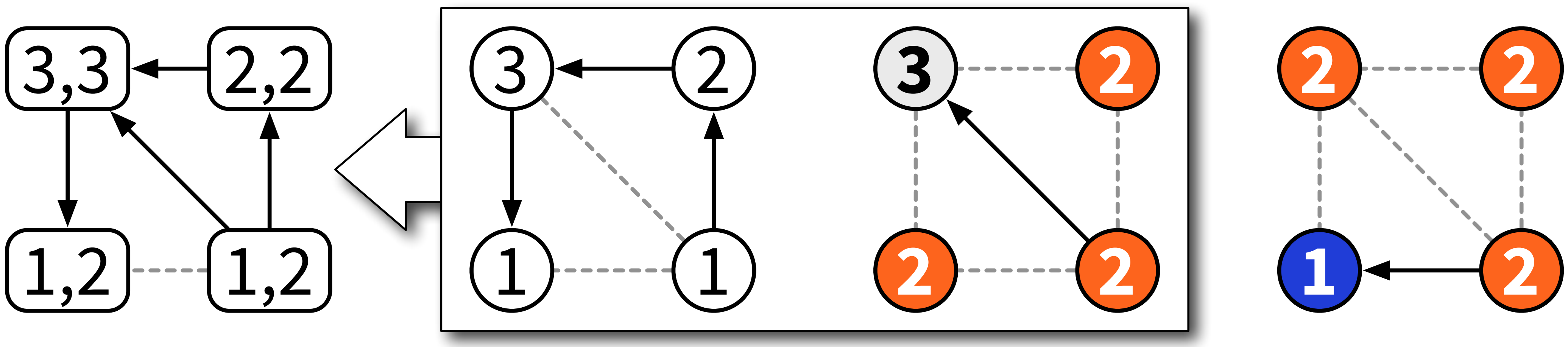


Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
  - add one forest → $3(\Delta + 1)$ colours



Algorithm BDColour:
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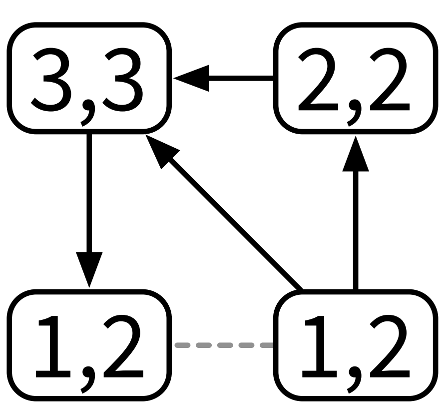
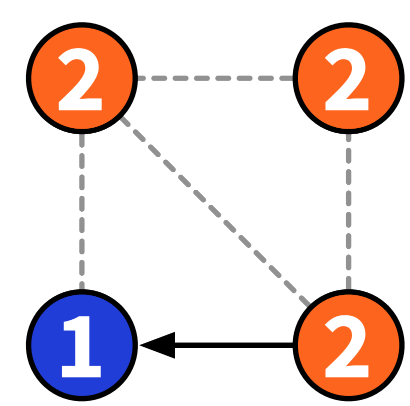
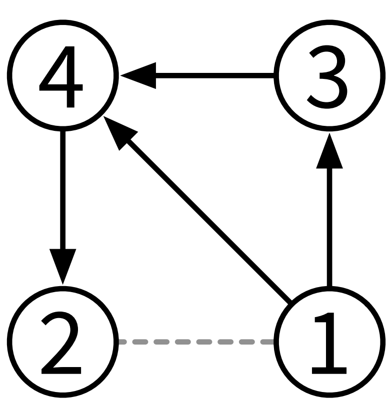
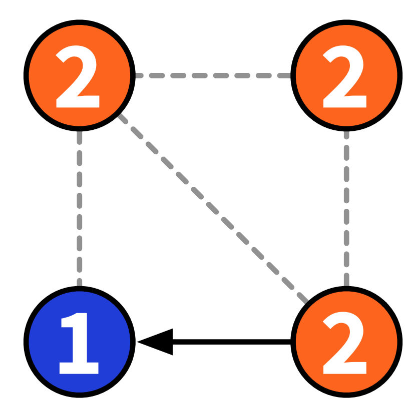




Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
  - add one forest → $3(\Delta + 1)$ colours → reduce





Algorithm BDColour: **Fast graph colouring**

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
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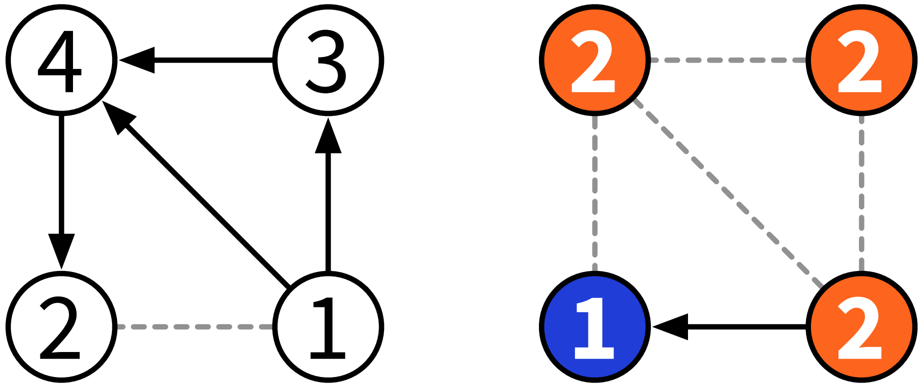

Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time $O(\Delta^2)$**
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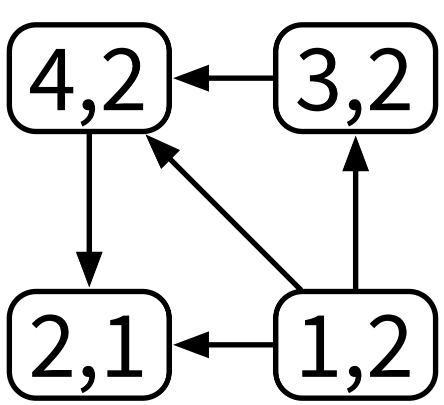
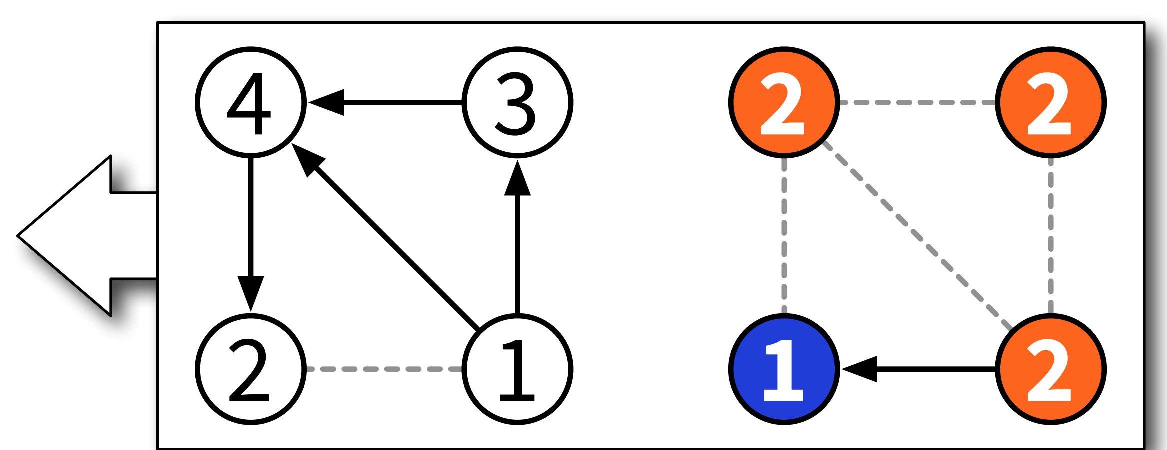
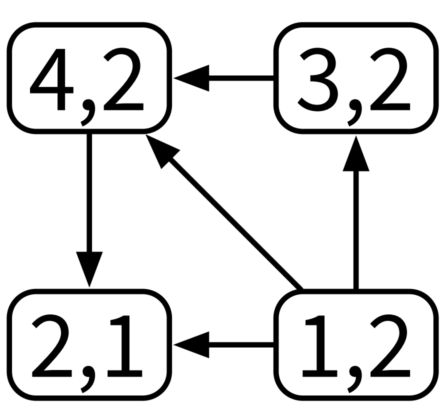




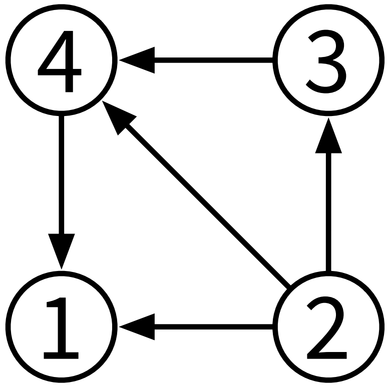
Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
  - add one forest → $3(\Delta + 1)$ colours



Algorithm BDColour:
Fast graph colouring

- **Merge pseudoforests in time *O*(Δ²)**
  - maintain colouring with Δ + 1 colours
  - add one forest → 3(Δ + 1) colours → reduce



Algorithm BDColour: **Fast graph colouring**

- **Merge pseudoforests in time $O(\Delta^2)$**
  - maintain colouring with $\Delta + 1$ colours
  - add one forest → $3(\Delta + 1)$ colours → reduce
- **Each merge + reduce takes $O(\Delta)$ rounds**
- **There are $O(\Delta)$ such steps**

Algorithm BDColour: **Fast graph colouring**

- **Unique identifiers → orientation**
- **Port numbers → partition edges in Δ directed pseudoforests**
- **3-colour pseudoforests in time $O(\log^* n)$**
- **Merge pseudoforests in time $O(\Delta^2)$**

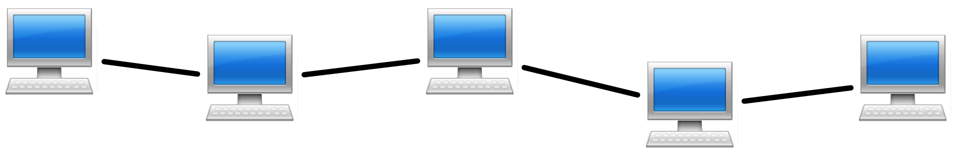
Summary:
LOCAL model

- **Unique identifiers**
- **Everything can be computed**
- **What can be computed fast?**
  - example: graph colouring

Summary:
LOCAL model

- **Unique identifiers**
- **Everything can be computed**
  - cheating with large messages
  - what if we can only use small messages?
  - this is covered next week…

- **Weeks 1–2: informal introduction**
  - network = path



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