Microeconomic Theory 1
Helsinki GSE – Fall 2019
Juuso Välimäki
TA: Michele Crescenzi

Problem Set 3 - Solutions

Problem 1

Let $T > 50$ be the time endowment. Leisure and income are denoted by ℓ and m, respectively. Under the old pay schedule, workers face a non-linear budget constraint. More specifically, in the (ℓ, m)-space, the budget line is $10\ell + m = 10T$ if $\ell \geq T - 40$, and $15\ell + m = 15(T - 40) + 400$ if $\ell < T - 40$. It is easy to see that there is a kink at $(T - 40, 400)$. With the new pay schedule, the budget constraint is linear and the budget line is simply $11\ell + m = 11T$.

Under the old schedule, workers choose the allocation $x^* = (T - 50, 550)$. Under the new schedule, x^* is still just affordable but cannot be optimal. To see why this is the case, notice that x^* satisfies

$$\frac{\partial u(x^*)/\partial \ell}{\partial u(x^*)/\partial m} = 15.$$

Since the new solution y^* must satisfy

$$\frac{\partial u(y^*)/\partial \ell}{\partial u(y^*)/\partial m} = 11, \tag{1}$$

we have that $x^* \neq y^*$. By standard arguments (see p. 54 in the textbook), you can also verify that $u(y^*) > u(x^*)$. In words, starting from x^*, it is feasible to move along the budget line towards better allocations (with less m and more ℓ) up until the new equilibrium condition (1) holds. Therefore, workers will earn less income under the new pay schedule, but at the same time they will consume a new leisure-income pair that is strictly preferred to the old one.

Problem 2

(a) Let us consider the optimization problem in the consumption space $\mathbb{R}^3_+$, where commodities are left shoes (L), right shoes (R), and all other goods (D). Since the number of *pairs* of shoes is $\min\{L, R\}$, the utility maximization problem is

$$\max_{L,R,D} \quad \min\{L, R\} \cdot D$$

subject to the budget constraint $p_L L + p_R R + D \le w$. It is straightforward to see that Walras' law holds, and that $L = R$ at every solution. Therefore, the optimization problem reduces to

$$\max_{S,D} \quad SD$$

subject to the budget constraint $(p_L + p_R)S + D = w$, where S is the quantity of pairs. You can easily verify that the unique solution to this problem is

$$(x_S(p,w), x_D(p,w)) = \left(\frac{w}{2(p_L+p_R)}, \frac{w}{2}\right),$$

and that the indirect utility function is

$$v(p,w) = u(x_S(p,w), x_D(p,w)) = \frac{w^2}{4(p_L+p_R)}.$$

(b) Everything is as in part (a) except for the budget constraint, which now becomes $p_L L + p_R R + D \le w + p_L L_0$. Define $\hat{w} := w + p_L L_0$. Following the same steps as in part (a), one obtains

$$(x_S(p,\hat{w}), x_D(p,\hat{w})) = \left(\frac{w + p_L L_0}{2(p_L+p_R)}, \frac{w + p_L L_0}{2}\right),$$

and the indirect utility function

$$v(p,\hat{w}) = \frac{(w + p_L L_0)^2}{4(p_L+p_R)}.$$

Notice that $x_S(p,\hat{w})$ is the *gross* demand for either kind of shoes. The net demand for left shoes is $x_S(p,\hat{w}) - L_0$.

An alternative way to find the Walrasian demands is the following. The utility function $u(S,D)$ is a monotone transformation of $u'(S,D) = S^{0.5}D^{0.5}$. The latter is a homogeneous function of degree one, and this implies that the Walrasian demand function is homogeneous of degree one in w (see Exercise 3.D.3 in the textbook). Thus it suffices to solve the optimization problem for $w = 1$ and, without any further calculations, one can easily conclude that $x(p,w) = w\,x(p,1)$ and $x(p,\hat{w}) = \hat{w}\,x(p,1)$. However, this reasoning does not apply to the indirect utility function. (Why?)

Problem 3

By indirect additivity,

$$\frac{\partial v^*(\boldsymbol{q})}{\partial q_i} = f'(\textstyle\sum_\ell v_\ell(q_\ell))\frac{\partial v_i(q_i)}{\partial q_i} \quad \text{for all } i. \tag{2}$$

By the envelope theorem,

$$\frac{\partial v^*(\boldsymbol{q})}{\partial q_i} = -\lambda x_i(\boldsymbol{q}) \quad \text{for all } i. \tag{3}$$

Combining (2) and (3), we obtain

$$\frac{\partial v_i(q_i)}{\partial q_i} x_j(\boldsymbol{q}) = \frac{\partial v_j(q_j)}{\partial q_j} x_i(\boldsymbol{q}) \quad \text{for all } i, j. \tag{4}$$

Differentiating both sides of (4) with respect to q_k (with $k \neq i, j$) yields

$$\frac{\partial v_i(q_i)}{\partial q_i} \frac{\partial x_j(\boldsymbol{q})}{\partial q_k} = \frac{\partial v_j(q_j)}{\partial q_j} \frac{\partial x_i(\boldsymbol{q})}{\partial q_k}. \tag{5}$$

Now, if we solve (4) for $\frac{\partial v_i(q_i)}{\partial q_i}$, substitute in (5), and rearrange, we get

$$\frac{\partial x_i(\boldsymbol{q})}{\partial q_k} \frac{1}{x_i(\boldsymbol{q})} = \frac{\partial x_j(\boldsymbol{q})}{\partial q_k} \frac{1}{x_j(\boldsymbol{q})}. \tag{6}$$

Finally, multiplying both sides of (6) by q_k yields the result.

Problem 4

(a) Fix two input prices vectors $w, w' \in \mathbb{R}^L_{++}$. We have:

$$\begin{aligned} c(w + w', q) &= (w + w') \cdot z(w + w', q) \\ &= w \cdot z(w + w', q) + w' \cdot z(w + w', q) \\ &\geq w \cdot z(w, q) + w' \cdot z(w', q) \\ &= c(w, q) + c(w', q), \end{aligned}$$

where the inequality follows from the fact that $z(w, q)$ and $z(w', q)$ are cost-minimizing at w and w', respectively.

(b) We need to show that $w \geq w'$ implies $c(w, q) \geq c(w', q)$. Suppose $w \geq w'$. Thus we have:

$$\begin{aligned} c(w, q) &= c((w - w') + w', q) \\ &\geq c(w - w', q) + c(w', q) \\ &\geq c(w', q), \end{aligned}$$

where the first inequality follows from the superadditivity of $c(w, q)$ with respect to w.

Problem 5

(a) The Lagrangian is

$$\mathcal{L} = w \cdot z + vx + \sum_{i=1}^{n} \lambda_i \left[g_i(q_i, z_i) - x \right].$$

Assuming an interior solution, the necessary Kuhn-Tucker conditions for optimality are

$$\begin{aligned} & v = \sum_{i=1}^{n} \lambda_i \\ & w_i + \lambda_i \frac{\partial g_i(q_i, z_i)}{\partial z_i} = 0 \quad \text{for all } i \\ & \lambda_i \left[g_i(q_i, z_i) - x \right] = 0, \quad \lambda_i \geq 0 \quad \text{for all } i. \end{aligned} \tag{7}$$

Notice that, in order for (7) to hold, we must have $\lambda_i > 0$ and $\frac{\partial g_i(q_i, z_i)}{\partial z_i} < 0$ for all i. Consequently, $g_i(q_i, z_i) = x$ for all i.

(b) We use x_i to denote the quantity of common factor that would solve the cost minimization problem of division i. Such a minimization problem is

$$\min_{z_i, x_i} \quad w_i z_i + \lambda_i x_i$$

subject to

$$x_i \geq g_i(q_i, z_i). \tag{8}$$

At an interior solution, the necessary Kuhn-Tucker conditions for optimality are

$$\begin{aligned} & \mu = \lambda_i \\ & w_i + \mu \frac{\partial g_i(q_i, z_i)}{\partial z_i} = 0 \\ & \mu \left[g_i(q_i, z_i) - x_i \right] = 0, \quad \mu \geq 0, \end{aligned}$$

where μ is the multiplier associated to the constraint (8). It is easy to verify that the result follows from the fact that $\mu = \lambda_i$.

(c) Since the production functions are not invertible, the functions $g_i(q_i, z_i)$ are not well-defined. The cost minimization problem becomes

$$\min_{z, x} \quad w \cdot z + vx$$

subject to

$$f_i(z_i, x) \geq q_i \quad \text{for all } i.$$

It is straightforward to verify that the solution is $z_i = q_i$ for all i, and $x = \max_i q_i$.

Problem 6

(a) The optimization problem is

$$\max_z \quad f(z)$$

subject to the constraint

$$pf(z) - w \cdot z \geq 0.$$

The Lagrangian is

$$\mathcal{L} = f(z) - \lambda \left[w \cdot z - pf(z) \right].$$

The Kuhn-Tucker necessary conditions for a maximizer z^* are

$$\frac{\partial f(z^*)}{\partial z_i} - \lambda w_i + \lambda p \frac{\partial f(z^*)}{\partial z_i} \leq 0 \text{ for all } i, \tag{9}$$

with equality if $z_i^* > 0$, and the complementary slackness condition

$$\lambda \left[w \cdot z^* - pf(z^*) \right] = 0, \quad \lambda \geq 0.$$

(b) Conditions (9) can be rewritten as

$$\lambda \left[w_i - p \frac{\partial f(z^*)}{\partial z_i} \right] \geq \frac{\partial f(z^*)}{\partial z_i} \text{ for all } i, \tag{10}$$

with equality if $z_i^* > 0$. Since marginal products are strictly positive by assumption, the right-hand side of (10) is strictly positive for all i. Together with the non-negativity constraint on λ, this implies that $\lambda > 0$ and that $w_i > p \frac{\partial f(z^*)}{\partial z_i}$ for all i. By complementary slackness, $\lambda > 0$ implies that the non-negativity constraint on profit is binding. Intuitively, if this constraint were not binding, then it would be feasible to purchase more production factors and, by the assumption of positive marginal products, produce more output.

(c) We have just seen in part (b) that the Lagrande multiplier λ is strictly positive. The interpretation is standard. It expresses the marginal change in the value function $f(z^*)$ as the non-negativity constraint on profit is relaxed. More specifically, rewrite the constraint in the more general form $w \cdot z - pf(z) \leq L$, where L is the largest loss that Joe can sustain. Then λ is the marginal change of the value function with respect to L.

(d) - (e) The supply function is the value function $f(z^*)$. By the envelope theorem,

$$\frac{\partial f(z^*)}{\partial p} = \lambda f(z^*) \geq 0 \qquad \text{and} \qquad \frac{\partial f(z^*)}{\partial w_i} = -\lambda < 0 \quad \text{for all } i.$$