Dynamic Models

Exercise 1. Suppose the population of a predator x_1 and its prey x_2 satisfy the following discrete dynamic equations

$$x_1(k+1) = x_1(k) - 0.8x_1(k) + 0.4x_2(k) + w_1(k), \tag{1}$$
$$x_2(k+1) = x_2(k) - 0.4x_1(k) + u(k) + w_2(k). \tag{2}$$

The first equation tells that an overcrowding of the predator population causes the predator population decrease, however the prey population causes the predator population to increase. Similarly for the second equation, the predator causes the prey population decrease, and an increases to internal food supply u at time k.

Task 1.1. *Write down last equations in the state space form.*

Task 1.2. *Let the population disturbance w_1, w_2 have variance $1, 2$, and $u(k) = 1$ for all k. With initial condition are set to $\bar{\mathbf{x}}(0) = [10,\ 20]^\top$, and $\mathbf{P}(0) = diag[40, 40]$, calculate the mean and covariance of both predator and prey populations at steady state, where $u = 1$. (Hint: at steady state $\bar{\mathbf{x}}(k) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k)$)*

Task 1.3. *Construct the dynamics of the covariance matrix $\mathbf{P}(k)$.*

Task 1.4. *Calculate the steady state covariance matrix $\mathbf{P}$ (Hint: use discrete time Lyapunov algorithm.)*

Exercise 2. Suppose a current that passes of RL circuit driven by a random voltage input $w(t)$ is desribed by the following dynamic equation

$$\dot{x} = -\frac{R}{L}x + w, \tag{3a}$$
$$\mathbb{E}[w(t)w(t+\tau)] = q_c\delta(\tau). \tag{3b}$$

Task 2.1. *Let us assume that the sampling time $\Delta t = t_k - t_{k-1}$ between two measurements are constant. Write down the discrete dynamic form of (3), so that you have the following form*

$$x(k+1) = Fx(k) + v(k)$$

where $v(k)$ is the noise component for discrete time. Find the covariance of $v(k)$

Exercise 3. Let $p(t) \in \mathbb{R}$, $v(t) \in \mathbb{R}$, and $a(t) \in \mathbb{R}$ be the position, velocity, and acceleration of some particle. Furthermore, the acceleration is controlled by some function $u(t)$ according to

$$a(t) = u(t). \tag{4}$$

Task 3.1. *Define the vector valued function*

$$y(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}. \tag{5}$$

Find a differential equation for $y(t)$ of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = Fy(t) + Bu(t), \tag{6}$$

where $F \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^2$.

Exercise 4. A particle is spinning around a circle of radius R in $\mathbb{R}^2$ with an angular velocity given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \omega(t). \tag{7}$$

Task 4.1. *Let $p(t) \in \mathbb{R}^2$ be position of the particle in Cartesian coordinates,*

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{8}$$

Find a differential equation for $p(t)$ in terms of $p(t)$ and $\omega(t)$.

Task 4.2. *The differential equation for $p(t)$ can be written as*

$$\frac{\mathrm{d}p}{\mathrm{d}t}(t) = F(\omega(t))p(t), \tag{9}$$

where $F(\omega(t)) \in \mathbb{R}^2$. What are the entries of $F(\omega(t))$?