Aalto University School of Science
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MS-E1280 Measure and integral, fall 2019

Homework assignment 1

Topics: Outer measures and measurable sets.

Deadline 11.11.2019 at 16:00.

1. Assume that $A \subset X$ is a finite or countable set and let $c : A \to [0, \infty]$. Define
$$\mu^* = \sum_{x \in A} c(x)\delta_x.$$
   (a) Show that μ^* is an outer measure. What sets are μ-measurable?
   (b) Show that μ^* is a finite outer measure if and only if $\sum_{x \in A} c(x) < \infty$.
   (c) Show that μ^* is a σ-finite outer measure if and only if $c(x) \in [0, \infty)$ for every $x \in A$.

2. Let μ^* be an outer measure on X and assume that $A \subset B \subset X$ are μ^*-measurable sets with $\mu^*(A) < \infty$. Show that $\mu^*(B \setminus A) = \mu^*(B) - \mu^*(A)$. Is the claim true if $\mu^*(A) = \infty$? Do the sets A and B have to be μ^*-measurable for the claim to hold?

3. Let μ^* be an outer measure on X with $\mu^*(X) < \infty$.
   (a) Show that if $A \subset X$ and $B \subset X$ is μ^*-measurable, then
$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) - \mu^*(A \cap B).$$
   (b) Show that if A, B, $C \subset X$ are μ^*-measurable, then
$$\begin{aligned}\mu^*(A \cup B \cup C) = \mu^*(A) &+ \mu^*(B) + \mu^*(C) \\ &- \mu^*(A \cap B) - \mu^*(A \cap C) - \mu^*(B \cap C) \\ &+ \mu^*(A \cap B \cap C).\end{aligned}$$
   (c) Find and prove the corresponding formula for the measure of a union of k sets.

   In what form the claims above hold true without the assumption $\mu^*(X) < \infty$?

4. Assume that μ and ν are measures on a σ-algebra $\mathcal{M}$ such that $\nu(A) \leq \mu(A)$ for every $A \in \mathcal{M}$.

   (a) Show that there exists a measure λ on $\mathcal{M}$ such that $\mu(A) = \nu(A) + \lambda(A)$ for every $A \in \mathcal{M}$.

   (b) Show that, in general, such a measure λ is not unique.

   (c) Show that such a measure λ is unique if ν is a σ-finite measure.

   Hint: $\lambda(A) = \sup\{\mu(B) - \nu(B) : B \subset A, \nu(B) < \infty, \ B \in \mathcal{M}\}$, $A \in \mathcal{M}$.

5. Assume that μ^* is an outer measure on X, A_r, $0 < r \leq 1$, μ^*-measurable sets satisfying $A_r \subset A_s$ for $r \leq s$ and $\mu^*(A_1) < \infty$. Prove that $A = \bigcap_{r>0} A_r$ is μ^*-measurable and

$$\mu^*(A) = \lim_{r \to 0} \mu^*(A_r).$$

6. Let $\mathcal{F}$ be a collection of subsets of X. The smallest σ-algebra of subsets of X containing $\mathcal{F}$ is called the σ-algebra generated by $\mathcal{F}$. Show that there exists a unique σ-algebra generated by $\mathcal{F}$.

   Hint: Show that the intersection of arbitrarily many σ-algebras is an σ-algebra.