Multivariate Location and Scatter
Lecture 2
M-estimates of Location and Scatter

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Outline

Moment Based Functionals

The mean vector and the covariance matrix are the most well-known location and scatter measures.

WHY DO WE NEED OTHER LOCATION AND SCATTER MEASURES???

Scatter Functionals

There are several other location and scatter functionals, even families of them, having different desirable properties (robustness, efficiency, limiting multivariate normality, fast computations, etc)

Moment Based Functionals

Location and scatter functionals can be based on the third and fourth moments as well. A location functional based on third moments is

$$T_2(F_x) = \frac{1}{p} E\left((x - E(x))^T Cov(F_x)^{-1}(x - E(x))x\right)$$

and a scatter matrix functional based on fourth moments is

$$S_2(F_x) = \frac{1}{p+2} E\left((x - E(x))(x - E(x))^T Cov(F_x)^{-1}(x - E(x))(x - E(x))^T\right).$$

Moment Based Functionals

The location functional $T_2(F_x)$ based on third moments and the scatter functional $S_2(F_x)$ based on fourth moments, together with the mean vector $T_1(F_x)$ and the covariance matrix $S_1(F_x)$, can be applied to construct measures of multivariate skewness and kurtosis, respectively. In the case of standard multivariate normal distribution $T_2(F_x) = 0_p$ and $S_2(F_x) = I_p$.

Location and Scatter under Symmetry Assumptions

Theorem

Under the assumption of central symmetry, all location functionals are equal to the center of symmetry.

Proof:...

Theorem

Under the assumption of multivariate elliptical distribution, all scatter functionals are proportional.

Proof:...

M-estimators of Location and Scatter

M-functionals

M-functionals of location and scatter are commonly used. They are defined as solutions of the two equations

$$T(F_x) = E(w_1(r))^{-1} E(w_1(r)x)$$

and

$$S(F_x) = E(w_2(r)(x - T(F_x))(x - T(F_x))^T),$$

where $w_1(r)$ and $w_2(r)$ are nonnegative continuous functions of the Mahalanobis distance $r = ||S(F_x)^{-1/2}(x - T(F_x))||$. (The $||\cdot||$ here denotes the l_2 norm of $\cdot$.)

M-functionals

The mean vector and the regular covariance matrix are M-functionals with $w_1(r) = w_2(r) = 1$, and as an other example, the Hettmansperger-Randles functionals have the weight functions

$$w_1(r) = \frac{1}{r} \quad \text{and} \quad w_2(r) = \frac{p}{r^2}.$$

Several other weight functions have been proposed in the literature.

One Step M-functionals

One Step M-functionals

Another important family of location and scatter functionals is the family of one step M-functionals. Given a pair of location and scatter functionals (T_1, S_1), the one step M-functionals are defined to be

$$T_2(F_x) = E(w_1(r_1))^{-1} E(w_1(r_1)x)$$

and

$$S_2(F_x) = E(w_2(r_1)(x - T_1(F_x))(x - T_1(F_x))^T),$$

where $w_1(r)$ and $w_2(r)$ are again nonnegative continuous weight functions and $r_1 = ||S_1(F_x)^{-1/2}(x - T_1(F_x))||$.

One Step M-functionals

The location functional based on third moments and the scatter functional based on fourth moments are obtained with choices $T_1(F_x) = E(x)$, $S_1(F_x) = E\left((x - E(x))(x - E(x))^T\right)$, $w_1(r) = r^2/p$ and $w_2(r) = r^2/(p+2)$.

References I

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H. Oja, Multivariate Nonparametric Methods With R, Springer-Verlag, New York (2010).