Aalto University School of Science
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MS-E1280 Measure and integral, fall 2019

Homework assignment 2
Topics: Outer measures and measurable sets.

Deadline 18.11.2019 at 16:00.

The problems 1–3 are related to Example 1.3 (7) in the lecture notes.

1. Let $\mathcal{F} \subset \{A : A \subset X\}$ be a collection of sets such that $\emptyset \in \mathcal{F}$ and there exist $B_i \in \mathcal{F}$, $i = 1, 2, \dots$, with $X = \bigcup_{i=1}^{\infty} B_i$. Let $\rho : \mathcal{F} \to [0, \infty]$ be a set function such that $\rho(\emptyset) = 0$. Show that $\mu^* : \{A : A \subset X\} \to [0, \infty]$ defined by
$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) : B_i \in \mathcal{F}, A \subset \bigcup_{i=1}^{\infty} B_i \right\},$$
for every $A \subset X$, is an outer measure on X.

2. (Continues) Let ρ be a monotone and countably subadditive set function on $\mathcal{F}$, that is, $\rho(B) \le \sum_{i=1}^{\infty} \rho(B_i)$ for every $B \subset \bigcup_{i=1}^{\infty} B_i$ with $B \in \mathcal{F}$ and $B_i \in \mathcal{F}$, $i = 1, 2, \dots$. Show that $\mu^* = \rho$ on $\mathcal{F}$.

3. (Continues) Assume that $\mathcal{F}$ is a σ-algebra and let $\rho : \mathcal{F} \to [0, \infty]$ be a countably additive set function on $\mathcal{F}$ (that is, ρ is a measure on $\mathcal{F}$).
   (a) Show that
   $$\mu^*(A) = \inf\{\mu^*(B) : B \in \mathcal{F},\ A \subset B\}$$
   for every $A \subset X$.
   (b) Show that every set in $\mathcal{F}$ is μ^*-measurable.
   (c) Show that μ^* is a regular outer measure, that is, for every set $A \subset X$ there exists a μ^*-measurable set $B \subset X$ such that $A \subset B$ and $\mu^*(B) = \mu^*(A)$.

4. This problem is related to Carathéodory's construction on $\mathbb{R}^n$, see Example 1.3 (8) in the lecture notes.

   (a) Show that Carathéodory's construction always gives an outer measure.

   (b) Show that Carathéodory's construction always gives a Borel measure.

   (c) If the members of the covering family in the construction are Borel sets, show that the measure is Borel regular.

5. Let μ^* and ν^* be Radon outer measures on $\mathbb{R}^n$.

   (a) Assume $\mu^*(G) = \nu^*(G)$ for every open set $G \subset \mathbb{R}^n$. Show that $\mu^*(E) = \nu^*(E)$ for every $E \subset \mathbb{R}^n$.

   (b) Assume $\mu^*(F) = \nu^*(F)$ for every closed set $F \subset \mathbb{R}^n$. Show that $\mu^*(E) = \nu^*(E)$ for every $E \subset \mathbb{R}^n$.

   (c) Assume $\mu^*(K) = \nu^*(K)$ for every compact set $K \subset \mathbb{R}^n$. Show that $\mu^*(E) = \nu^*(E)$ for every $E \subset \mathbb{R}^n$.

6. Let μ^* be a Borel outer measure on $\mathbb{R}^n$. Show that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

   whenever $A, B \subset \mathbb{R}^n$ with $\operatorname{dist}(A, B) > 0$.