Multivariate Location and Scatter
Lecture 4
Spatial Sign and Rank Based Functionals

Pauliina Ilmonen

Outline

Spatial Signs and Spatial Ranks

Spatial Signs and Spatial Ranks

Spatial Signs and Spatial Ranks

Spatial signs and spatial ranks are generalizations of univariate signs and ranks. Spatial signs and spatial ranks can be used, among other things, for constructing multivariate measures of location and scatter.

Spatial Signs and Spatial Ranks

Let x denote a p-variate random vector with a cumulative distribution function F_x and let $X = [x_1 \dots . x_n]$, where $x_1, ..., x_n$ are i.i.d. observations from the distribution F_x.

Spatial Sign

The spatial sign $U(x_i)$ is defined as

$$U(x_i) = \begin{cases} x_i(||x_i||)^{-1} & x_i \neq 0, \\ 0 & x_i = 0. \end{cases}$$

Spatial Rank

The spatial rank $R(x_i)$ is defined as

$$R(x_i) = \frac{1}{n}\sum_{j=1}^{n} U(x_i - x_j).$$

Spatial Signed Rank

The spatial signed rank $Q(x_i)$ is defined as

$$Q(x_i) = \frac{1}{2n} \sum_{j=1}^{n} (U(x_i - x_j) + U(x_i + x_j)).$$

Theoretical Functionals

The theoretical spatial sign $U(F_x)$, spatial rank $R(F_x)$, and spatial signed rank $Q(F_x)$ functionals are defined as

$$U(F_x) = E(U(x)), \; R(F_x) = E(R(x)), \text{ and } Q(F_x) = E(Q(x)).$$

Spatial Median

Let

$$D_n(\mu) = \frac{1}{n}\sum_{i=1}^{n} ||x_i - \mu|| - ||x_i||,$$

and let

$$D(\mu) = E(||x_i - \mu|| - ||x_i||).$$

Now the sample spatial median is defined as the minimizer $\mu(X)$ of $D_n(\mu)$, and the population spatial median is defined as the minimizer $\mu(F_x)$ of $D(\mu)$.

Affine Equivariant Spatial Median

Let $S = S(X)$ denote any scatter matrix. Then the transformation retransformation spatial median $T_r(X)$ is defined as

$$T_r(X) = S(X)^{1/2}\mu(XS(X)^{-1/2}).$$

Spatial Sign and Rank Covariance Matrices

The spatial Kendall's tau covariance matrix functional $S_T(F_x)$, and the spatial rank covariance matrix functional $S_R(F_x)$ are defined as

$$S_T(F_x) = E(U(x_1 - x_2)(U(x_1 - x_2))^T),$$

and

$$S_R(F_x) = E(R(x)(R(x))^T),$$

where x_1 and x_2 are independent observations from F_x.

Spatial Sign and Rank Covariance Matrices

The spatial Kendall's tau sample matrix $S_T(X)$ and the sample spatial rank covariance matrix $S_R(X)$ are defined as

$$S_T(X) = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} U(x_i - x_j)(U(x_i - x_j))^T,$$

and

$$S_R(X) = \frac{1}{n} \sum_{i=1}^{n} R(x_i)(R(x_i))^T.$$

References

H. Oja, Multivariate Nonparametric Methods With R, Springer-Verlag, New York (2010).