



**Aalto University**  
School of Electrical  
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# **ELEC-E8740 — Discretization of Continuous-Time Dynamic Models**

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# Contents

- 1 Intended Learning Outcomes and Recap
- 2 Discretization of Linear ODEs
- 3 Discretization of Linear SDEs
- 4 Discretization of Non-Linear Systems
- 5 Summary

# Intended Learning Outcomes

After this lecture, you will be able to:

- explain why continuous-time dynamic models need to be discretized in practice
- construct discrete-time dynamic models from linear ODE and SDE state-space models
- construct approximate discrete-time dynamic models from non-linear ODE and SDE models

## Recap (1)

- Higher order ODEs and SDEs can be transformed to a first-order vector-valued equation system
- The deterministic linear state-space model is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$
$$\mathbf{y}_n = \mathbf{G}\mathbf{x}_n + \mathbf{r}_n$$

- The stochastic linear state-space model with stochastic input process  $\mathbf{w}(t)$  is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w\mathbf{w}(t)$$
$$\mathbf{y}_n = \mathbf{G}\mathbf{x}_n + \mathbf{r}_n$$

- The 2D Wiener velocity model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

## Recap (2)

- Nonlinear continuous-time state-space model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t) \\ \mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n\end{aligned}$$

- Linear discrete-time state-space model:

$$\begin{aligned}\mathbf{x}_n &= \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q\mathbf{q}_n \\ \mathbf{y}_n &= \mathbf{G}\mathbf{x}_n + \mathbf{r}_n\end{aligned}$$

- Nonlinear discrete-time state-space model:

$$\begin{aligned}\mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{B}_q(\mathbf{x}_{n-1})\mathbf{q}_n \\ \mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n\end{aligned}$$

# Discretization of Continuous-Time Models: Why?

- Sensor fusion is implemented in digital computers
- Data is only processed at  $t_1, t_2, \dots, t_n$
- Discretized continuous-time models are closely related to discrete-time models
- Example: Vehicle tracking

Discretization of continuous-time models is equivalent to solving the ODE/SDE model between  $t_{n-1}$  and  $t_n$

# Solving Linear First Order ODEs (1/2)

- Goal: Solve the first order ODE

$$\dot{x}(t) = ax(t) + bu(t),$$

on the interval  $(t_{n-1}, t_n]$ .

- Ansatz: Multiply by the **integrating factor**  $e^{-at}$

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) + e^{-at}bu(t)$$

i.e.

$$e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t)$$

- We can then identify the derivative on the left hand side:

$$\frac{d}{dt} [e^{-at}x(t)] = e^{-at}\dot{x}(t) - e^{-at}ax(t).$$

- Thus we have

$$\frac{d}{dt} [e^{-at}x(t)] = e^{-at}bu(t).$$

## Solving Linear First Order ODEs (2/2)

- We can now integrate the both sides:

$$\int_{t_{n-1}}^{t_n} \frac{d}{dt} [e^{-at}x(t)] dt = \int_{t_{n-1}}^{t_n} e^{-at}bu(t)dt$$

- Solution:

$$e^{-at_n}x(t_n) - e^{-at_{n-1}}x(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-at}bu(t)dt$$

- Rearranged:

$$x(t_n) = e^{a(t_n-t_{n-1})}x(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{a(t_n-t)}bu(t)dt$$

- Defining  $\Delta t = t_n - t_{n-1}$  this is

$$x(t_n) = e^{a\Delta t}x(t_{n-1}) + \int_0^{\Delta t} e^{a(\Delta t-t)}bu(t)dt$$



# Vector-Valued Linear First Order ODE

- General linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

- This is a vector-valued first order ODE

What is the integrating factor for vector-valued first order ODEs?

# Matrix Exponential

- Definition of the exponential:

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k$$

- Definition of the **matrix exponential**:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k$$

- Derivative of matrix exponential w.r.t. scalar  $t$ :

$$\frac{d}{dt} e^{\mathbf{A}t} = e^{\mathbf{A}t} \mathbf{A}$$

- Matrix exponential of  $\mathbf{A}^T$ :

$$(e^{\mathbf{A}})^T = e^{\mathbf{A}^T}$$

# Solving Linear First Order Vector ODEs (1/2)

- General linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

- Multiplication by the integrating factor  $e^{-\mathbf{A}t}$ :

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

- Rearranging:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

- Substituting  $\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)$ :

$$\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

## Solving Linear First Order Vector ODEs (2/2)

- We now have the ODE:

$$\frac{d}{dt} e^{-\mathbf{A}t} \mathbf{x}(t) = e^{-\mathbf{A}t} \mathbf{B}_u \mathbf{u}(t)$$

- Integration w.r.t.  $t$ :

$$\int_{t_{n-1}}^{t_n} d \left[ e^{-\mathbf{A}t} \mathbf{x}(t) \right] = \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t} \mathbf{B}_u \mathbf{u}(t) dt$$

$$\left[ e^{-\mathbf{A}t} \mathbf{x}(t) \right]_{t=t_{n-1}}^{t_n} = \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t} \mathbf{B}_u \mathbf{u}(t) dt$$

$$e^{-\mathbf{A}t_n} \mathbf{x}(t_n) - e^{-\mathbf{A}t_{n-1}} \mathbf{x}(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t} \mathbf{B}_u \mathbf{u}(t) dt$$

- Rearranging:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})} \mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathbf{u}(t) dt$$

# Zero-Order-Hold Inputs

- Solution:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})} \mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathbf{u}(t) dt,$$

- The input  $\mathbf{u}(t)$  can be often assumed to be constant between sampling instants (zero-order-hold; ZOH)
- Then:

$$\int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathbf{u}(t) dt = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u dt \mathbf{u}(t_{n-1})$$

# Discretized Deterministic Linear Dynamic Model

- Linear continuous-time dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

- Discretized dynamic model:

$$\mathbf{x}_n = \mathbf{F}_n\mathbf{x}_{n-1} + \mathbf{L}_n\mathbf{u}_{n-1},$$

where

$$\mathbf{F}_n \triangleq e^{\mathbf{A}(t_n - t_{n-1})}$$

$$\mathbf{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u dt$$

The discretized dynamic model is completely equivalent to the continuous-time model

# Example: Deterministic 1D Motion Model (1/4)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Recall:

$$\mathbf{F}_n = e^{\mathbf{A}\Delta t} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j (\Delta t)^j$$

## Example: Deterministic 1D Motion Model (2/4)

- Powers of  $\mathbf{A}$ :

$$\mathbf{A}^0 = \mathbf{I}$$

$$\mathbf{A}^1 = \mathbf{A}$$

$$\mathbf{A}^j = \mathbf{0} \quad j \geq 2$$

- Hence:

$$\begin{aligned} \mathbf{F}_n &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j (\Delta t)^j = \frac{1}{0!} \mathbf{I} (\Delta t)^0 + \frac{1}{1!} \mathbf{A} \Delta t \\ &= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \end{aligned}$$



## Example: Deterministic 1D Motion Model (3/4)

- Input matrix:

$$\mathbf{L}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_u dt$$

where:

$$e^{\mathbf{A}(t_n-t)} = \begin{bmatrix} 1 & t_n - t \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Example: Deterministic 1D Motion Model (4/4)

- Continuous-time model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Discretized model:

$$\mathbf{x}_n = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{n-1} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \mathbf{u}_{n-1}$$

# Stochastic Linear Dynamic Model

- Stochastic linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w\mathbf{w}(t),$$

- The only difference to the deterministic model is the input  $\mathbf{u}(t)$  ( $\mathbf{w}(t)$ )
- $\mathbf{w}(t)$  is a zero-mean white stochastic process
- Auto-correlation function:

$$R_{ww}(\tau) = E\{\mathbf{w}(t + \tau)\mathbf{w}(t)\} = \mathbf{\Sigma}_w\delta(\tau)$$

- Here  $\mathbf{\Sigma}_w$  is the spectral density of the while noise.
- Hence:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})}\mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)}\mathbf{B}_w\mathbf{w}(t)dt$$

# Integration of the Stochastic Process

- Model:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})} \mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) dt$$

- $\mathbf{w}(t)$  is stochastic; not ZOH and not even integrable (with standard tools)
- Define a random variable as the **process noise**:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) dt$$

- Then:

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{q}_n$$

# Mean of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt.$$

- We get

$$\begin{aligned} E\{\mathbf{q}_n\} &= E\left\{ \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt \right\} \\ &= \int_{t_{n-1}}^{t_n} E\left\{ e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) \right\} dt \\ &= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w E\{\mathbf{w}(t)\} dt \\ &= 0. \end{aligned}$$

# Covariance of the Process Noise (1/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt$$

- Covariance:

$$\text{Cov}\{\mathbf{q}_n\}$$

$$= E\{(\mathbf{q}_n - E\{\mathbf{q}_n\})(\mathbf{q}_n - E\{\mathbf{q}_n\})^T\}$$

$$= E\{\mathbf{q}_n \mathbf{q}_n^T\}$$

$$= E\left\{ \left( \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt \right) \left( \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w \mathbf{w}(\tau) d\tau \right)^T \right\}$$

$$= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w E\{\mathbf{w}(t)\mathbf{w}(\tau)^T\} \mathbf{B}_w^T (e^{\mathbf{A}(t_n-\tau)})^T d\tau dt \dots$$

## Covariance of the Process Noise (2/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt$$

- Covariance:

$$\begin{aligned} \text{Cov}\{\mathbf{q}_n\} &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w E \left\{ \mathbf{w}(t) \mathbf{w}(\tau)^T \right\} \mathbf{B}_w^T (e^{\mathbf{A}(t_n-\tau)})^T d\tau dt \\ &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w R_{ww}(t-\tau) \mathbf{B}_w^T (e^{\mathbf{A}(t_n-\tau)})^T d\tau dt \\ &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \Sigma_w \delta(t-\tau) \mathbf{B}_w^T (e^{\mathbf{A}(t_n-\tau)})^T d\tau dt \\ &= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w \Sigma_w \mathbf{B}_w^T e^{\mathbf{A}^T(t_n-\tau)} d\tau \end{aligned}$$

# Properties of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt$$

- Mean and covariance:

$$E\{\mathbf{q}_n\} = 0$$

$$\text{Cov}\{\mathbf{q}_n\} = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^T e^{\mathbf{A}^T(t_n-\tau)} d\tau \triangleq \mathbf{Q}_n$$

- Distribution:

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$$



# Discretized Stochastic Linear Dynamic Model

- Discretized stochastic dynamic model:

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{q}_n$$

where:

$$\mathbf{F}_n \triangleq e^{\mathbf{A}(t_n - t_{n-1})}$$

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$$

$$\mathbf{Q}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^T e^{\mathbf{A}^T(t_n - \tau)} d\tau$$

The discretized stochastic dynamic model is completely equivalent to the continuous-time model

# Example: 1D Wiener Velocity Model (1/3)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

with white noise process  $w(t)$  and  $R_{ww}(\tau) = \sigma_w^2 \delta(\tau)$

- Process noise covariance:

$$\mathbf{Q}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w \Sigma_w \mathbf{B}_w^T e^{\mathbf{A}^T(t_n-\tau)} d\tau$$

- Recall:

$$e^{\mathbf{A}(t_n-\tau)} = \begin{bmatrix} 1 & t_n - \tau \\ 0 & 1 \end{bmatrix}$$
$$e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w = \begin{bmatrix} 1 & t_n - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix}$$

## Example: 1D Wiener Velocity Model (2/3)

- Process noise covariance:

$$\begin{aligned} \mathbf{Q}_n &= \int_{t_{n-1}}^{t_n} \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix} \sigma_w^2 \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix}^T d\tau \\ &= \sigma_w^2 \int_{t_{n-1}}^{t_n} \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix} \begin{bmatrix} t_n - \tau & 1 \end{bmatrix} d\tau \\ &= \sigma_w^2 \int_{t_{n-1}}^{t_n} \begin{bmatrix} (t_n - \tau)^2 & t_n - \tau \\ t_n - \tau & 1 \end{bmatrix} d\tau \\ &= \sigma_w^2 \begin{bmatrix} -\frac{1}{3}(t_n - \tau)^3 & -\frac{1}{2}(t_n - \tau)^2 \\ -\frac{1}{2}(t_n - \tau)^2 & \tau \end{bmatrix}_{\tau=t_{n-1}}^{t_n} \\ &= \sigma_w^2 \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix} \end{aligned}$$

## Example: 1D Wiener Velocity Model (3/3)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

- Discretized model:

$$\begin{bmatrix} p_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ v_{n-1} \end{bmatrix} + \mathbf{q}_n$$

with  $\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$  and

$$\mathbf{Q}_n = \sigma_w^2 \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix}$$

# Discretization of Nonlinear Dynamic Models

- Objective: Discretization of nonlinear models

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

and

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Problem: In most cases, no exact approach exists
- A few possible approaches:
  - Linearization of the nonlinear model followed by discretization
  - Approximation of the derivative (integral)
  - Exact integration (of at least the dynamics)
  - & many more...

# Linearization of Nonlinear Models

- Nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

- 1st order Taylor series approximation of  $\mathbf{f}(\mathbf{x}(t))$  around  $\mathbf{x}(t) = \mathbf{x}(t_{n-1})$ :

$$\mathbf{f}(\mathbf{x}(t)) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1})$$

- Approximation of the ODE:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_u\mathbf{u}(t)$$

# Discretization of Linearized Models (1/2)

- Approximation of the ODE:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_u \mathbf{u}(t)$$

- Rewritten approximation of the ODE

$$\dot{\mathbf{x}}(t) \approx \mathbf{A}_x \mathbf{x}(t) + \mathbf{f}(\mathbf{x}_{n-1}) - \mathbf{A}_x \mathbf{x}_{n-1} + \mathbf{B}_u \mathbf{u}(t)$$

- Solution of the approximation:

$$\begin{aligned} \mathbf{x}_n \approx & e^{\mathbf{A}_x \Delta t} \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{f}(\mathbf{x}_{n-1}) \\ & - \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{A}_x \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} \mathbf{B}_u \mathbf{u}(t) dt \end{aligned}$$

# Discretization of Linearized Models (2/2)

- Solution of the approximation:

$$\begin{aligned}\mathbf{x}_n \approx & e^{\mathbf{A}_x \Delta t} \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} d\mathbf{f}(\mathbf{x}_{n-1}) \\ & - \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{A}_x \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} \mathbf{B}_u \mathbf{u}(t) dt\end{aligned}$$

- Simplified solution:

$$\mathbf{x}_n \approx \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} d\mathbf{f}(\mathbf{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} \mathbf{B}_u \mathbf{u}(t) dt$$



# Discretization of Linearized Models (Stochastic)

- Stochastic nonlinear model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Discretization is the same as for the ODE model:

$$\begin{aligned}\mathbf{x}_n &\approx \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{f}(\mathbf{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} \mathbf{B}_w \mathbf{w}(t) dt \\ &= \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n\end{aligned}$$

with

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n),$$

$$\mathbf{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-\tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^T e^{\mathbf{A}_x^T(t_n-\tau)} d\tau$$

# Properties of the Discretization

- Stochastic nonlinear model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Linearized model:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_w\mathbf{w}(t)$$

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Integration is exact, model is not
- Discretization is not exact
- Linearization is local, may cause problems

# Example: Quasi-Constant Turn Model (1/5)

- Model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t) \cos(\varphi(t)) \\ v(t) \sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(t)$$

- Jacobian of  $\mathbf{f}(\mathbf{x}(t))$ :

$$\begin{aligned} \mathbf{A}_x &= \begin{bmatrix} 0 & 0 & \cos(\varphi(t)) & -v(t) \sin(\varphi(t)) \\ 0 & 0 & \sin(\varphi(t)) & v(t) \cos(\varphi(t)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cos(\varphi_{n-1}) & -v_{n-1} \sin(\varphi_{n-1}) \\ 0 & 0 & \sin(\varphi_{n-1}) & v_{n-1} \cos(\varphi_{n-1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

## Example: Quasi-Constant Turn Model (2/5)

- Powers of  $\mathbf{A}_x$ :

$$\mathbf{A}_x^0 = \mathbf{I}$$

$$\mathbf{A}_x^1 = \mathbf{A}_x$$

$$\mathbf{A}_x^2 = \mathbf{0}$$

- Matrix exponential:

$$e^{\mathbf{A}_x(t_n-t)} = \mathbf{I} + \mathbf{A}_x(t_n - t)$$

$$= \begin{bmatrix} 1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1} \sin(\varphi_{n-1})(t_n - t) \\ 0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1} \cos(\varphi_{n-1})(t_n - t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Example: Quasi-Constant Turn Model (3/5)

- Integral:

$$\begin{aligned}
 & \int_{t_{n-1}}^{t_n} \begin{bmatrix} 1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1} \sin(\varphi_{n-1})(t_n - t) \\ 0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1} \cos(\varphi_{n-1})(t_n - t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} dt \\
 &= \begin{bmatrix} t & 0 & -\frac{(t_n-t)^2}{2} \cos(\varphi_{n-1}) & \frac{(t_n-t)^2}{2} v_{n-1} \sin(\varphi_{n-1}) \\ 0 & t & -\frac{(t_n-t)^2}{2} \sin(\varphi_{n-1}) & -\frac{(t_n-t)^2}{2} v_{n-1} \cos(\varphi_{n-1}) \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \Bigg|_{t=t_{n-1}}^{t_n} \\
 &= \begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2} \cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2} v_{n-1} \sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2} \sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2} v_{n-1} \cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix}
 \end{aligned}$$

## Example: Quasi-Constant Turn Model (4/5)

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} dt \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Second term:

$$\begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2} \cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2} v_{n-1} \sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2} \sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2} v_{n-1} \cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix} \begin{bmatrix} v_{n-1} \cos(\varphi_{n-1}) \\ v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \Delta t v_{n-1} \cos(\varphi_{n-1}) \\ \Delta t v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$

## Example: Quasi-Constant Turn Model (5/5)

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} d\mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Discretization of Linearized Model:

$$\begin{bmatrix} p_n^x \\ p_n^y \\ v_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} p_{n-1}^x \\ p_{n-1}^y \\ v_{n-1} \\ \varphi_{n-1} \end{bmatrix} + \begin{bmatrix} \Delta t v_{n-1} \cos(\varphi_{n-1}) \\ \Delta t v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix} + \mathbf{q}_n$$

- What about  $\mathbf{Q}_n$ ?

$$\mathbf{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-\tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^T e^{\mathbf{A}_x^T(t_n-\tau)} d\tau$$

# Euler Approximation

- Dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

- Integral equation:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} \mathbf{f}(\mathbf{x}(t))dt + \int_{t_{n-1}}^{t_n} \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)dt$$

- Idea: Approximate the integral rather than the model
- Euler approximation:

$$\mathbf{x}_n \approx \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \Delta t \mathbf{B}_u(\mathbf{x}_{n-1})\mathbf{u}_{n-1}.$$



# Euler–Maruyama Discretization (1)

- Stochastic dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Integral representation:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} \mathbf{f}(\mathbf{x}(t))dt + \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)dt$$

- Process noise definition:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)dt$$

# Mean of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \mathbf{w}(t) dt$$

- Mean:

$$\begin{aligned} E\{\mathbf{q}_n\} &= E\left\{ \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \mathbf{w}(t) dt \right\} \\ &= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) E\{\mathbf{w}(t)\} dt \\ &= 0 \end{aligned}$$

# Covariance of the Process Noise (1/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \mathbf{w}(t) dt$$

- Covariance:

$$\begin{aligned} \text{Cov}\{\mathbf{q}_n\} &= E \left\{ \left( \int_{t_{n-1}}^{t_n} \mathbf{B}_w \mathbf{w}(t) dt \right) \left( \int_{t_{n-1}}^{t_n} \mathbf{B}_w \mathbf{w}(\tau) d\tau \right)^T \right\} \\ &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) E\{\mathbf{w}(t)\mathbf{w}(\tau)^T\} \mathbf{B}_w(\mathbf{x}(t))^T d\tau dt \\ &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \boldsymbol{\Sigma}_w \delta(t - \tau) \mathbf{B}_w(\mathbf{x}(t))^T d\tau dt \\ &= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \boldsymbol{\Sigma}_w \mathbf{B}_w^T(\mathbf{x}(t)) dt \end{aligned}$$

## Covariance of the Process Noise (2/2)

- Covariance:

$$\text{Cov}\{\mathbf{q}_n\} = \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}(t))^T d\tau$$

- Rectangle approximation of the integral:

$$\begin{aligned} \text{Cov}\{\mathbf{q}_n\} &= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \boldsymbol{\Sigma}_w \mathbf{B}_w^T(\mathbf{x}(t)) d\tau \\ &\approx \mathbf{B}_w(\mathbf{x}_{n-1}) \boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^T (t_n - t_{n-1}) \\ &= \Delta t \mathbf{B}_w(\mathbf{x}_{n-1}) \boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^T \\ &\triangleq \mathbf{Q}_n \end{aligned}$$

## Euler–Maruyama Discretization (2)

- Dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Euler–Maruyama discretization:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

with  $\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$ ,  $\mathbf{Q}_n \approx \Delta t \mathbf{B}_w(\mathbf{x}_{n-1}) \boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\top$

- ... or equivalently:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \sqrt{\Delta t} \mathbf{B}_w(\mathbf{x}_{n-1}) \mathbf{q}_n$$

with  $\mathbf{q}_n \sim \mathcal{N}(0, \boldsymbol{\Sigma}_w)$

- Discretization is not exact

## Summary (1/3)

- The discretization of the linear ODE model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

is

$$\mathbf{x}_n = \mathbf{F}_n\mathbf{x}_{n-1} + \mathbf{L}_n\mathbf{u}_{n-1}$$

$$\mathbf{F}_n \triangleq e^{\mathbf{A}(t_n - t_{n-1})}, \quad \mathbf{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u dt$$

- The discretization of the linear SDE model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w\mathbf{w}(t)$$

is

$$\mathbf{x}_n = \mathbf{F}_n\mathbf{x}_{n-1} + \mathbf{q}_n, \quad \mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$$

$$\mathbf{Q}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^T e^{\mathbf{A}^T(t_n - \tau)} d\tau$$

## Summary (2/3)

- Nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Discretization of the linearized model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w\mathbf{w}(t) \\ &\approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_w\mathbf{w}(t)\end{aligned}$$

↓

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-t)} d\mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

with

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n), \quad \mathbf{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n-\tau)} \mathbf{B}_w \Sigma_w \mathbf{B}_w^T e^{\mathbf{A}_x^T(t_n-\tau)} d\tau$$

## Summary (3/3)

- Euler–Maruyama discretization:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

↓

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

with

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n),$$

$$\mathbf{Q}_n \approx \Delta t \mathbf{B}_w(\mathbf{x}_{n-1}) \boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\top.$$