Aalto University School of Science
Department of Mathematics and Systems Analysis Kinnunen

MS-E1280 Measure and integral, fall 2019

Homework assignment 3

Topics: Lebesgue outer measure and measurable sets.

Deadline 25.11.2019 at 16:00.

1. (a) Show that closed intervals in the definition of the Lebesgue outer measure can be replaced by open intervals, that is,

$$m^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{vol}(J_i) : A \subset \bigcup_{i=1}^{\infty} J_i\right\},$$

   for every $A \subset \mathbb{R}^n$, where the infimum is taken over all coverings of A by countably many open intervals J_i, $i = 1, 2, \ldots$.

   (b) Show that for a compact set, it is possible to use finite coverings in the definition of the Lebesgue measure.

2. Show that $A \subset \mathbb{R}^n$ is Lebesgue measurable if and only if

$$m^*(Q) = m^*(Q \cap A) + m^*(Q \setminus A)$$

   for every cube Q in $\mathbb{R}^n$.

3. Let $A \subset \mathbb{R}^n$. Show that the following claims are equivalent:

   (a) A is Lebesgue measurable,

   (b) A is a $\mathcal{G}_\delta$ set with a set of measure zero removed,

   (c) A is a union of a $\mathcal{F}_\sigma$ set and a set of measure zero.

4. (a) Assume that $A \subset \mathbb{R}^n$ is such that $m^*(\partial A) = 0$. Show that A is Lebesgue measurable.

   (b) Give an example of a Lebesgue measurable set $A \subset \mathbb{R}^n$ with $m^*(\partial A) > 0$.

5. Let $A \subset \mathbb{R}^n$ and denote $\delta A = \{\delta x \in \mathbb{R}^n : x \in A\}$ with $\delta > 0$.

   (a) Show that A is Lebesgue measurable if and only if δA is Lebesgue measurable.

   (b) Show that $m^*(\delta A) = \delta^n m^*(A)$ for every $A \subset \mathbb{R}^n$.

6. Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set with $m^*(A) = \infty$. Show that for every t, $0 \le t < \infty$, there exists a Lebesgue measurable set $A_t \subset A$ such that $m^*(A_t) = t$.