Solutions to Assignment 1

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1.a

- Monotonicity. Let $B_1 \subset B_2$. Clearly for each x the measure δ_x is monotonous, i.e. $\delta_x(B_1) \leq \delta_x(B_2)$. Since for all x we have $c(x) \geq 0$ we obtain that also

$$\mu^*(B_1) \leq \mu^*(B_2).$$

- σ-additivity. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of subsets of X. We first prove the subadditivity of all the δ_x. If $x \notin \bigcup_{i\in\mathbb{N}} A_i$ then

$$\delta_x(\bigcup_{i\in\mathbb{N}} A_i) = 0 \leq \sum_{i\in\mathbb{N}} \delta_x(A_I).$$

If $x \in \bigcup_{i\in\mathbb{N}} A_i$ then there is an i_0 with $x \in A_{i_0}$. Thus

$$\delta_x(\bigcup_{i\in\mathbb{N}} A_i) = 1 = \delta_x(A_{i_0}) \leq \sum_{i\in\mathbb{N}} \delta_x(A_i).$$

Now for μ^* we have

$$\mu^*(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{x\in A} c(x)\delta_x(\bigcup_{i\in\mathbb{N}} A_i)$$

and by the σ-subadditivity of δ_x

$$\leq \sum_{x\in A} c(x) \sum_{i\in\mathbb{N}} \delta_x(A_i)$$

and since all summands are nonnegative we can change the order of (infinite) summation

$$= \sum_{i\in\mathbb{N}} \sum_{x\in A} c(x)\delta_x(A_i)$$
$$= \sum_{i\in\mathbb{N}} \mu^*(A_i)$$

- <u>All subsets $B \subset X$ are measurable.</u> We have to prove that for each set E we have

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \setminus B).$$

First we prove that for each δ_x. If $x \notin E$ then

$$\delta_x(E) = 0 = \delta_x(E \cap B) + \delta_x(E \setminus B).$$

If $x \in E$ then x is in exactly one of $E \cap B$ and $E \setminus B$. Hence

$$\delta_x(E) = 1 = \delta_x(E \cap B) + \delta_x(E \setminus B).$$

Now for μ^* we have

$$\begin{aligned}\mu^*(E) &= \sum_{x \in A} c(x)\delta_x(E) \\ &= \sum_{x \in A} c(x)(\delta_x(E \cap B) + \delta(E \setminus B)) \\ &= \sum_{x \in A} c(x)\delta_x(E \cap B) + \sum_{x \in A} c(x)\delta(E \setminus B) \\ &= \mu^*(E \cap B) + \mu^*(E \setminus B).\end{aligned}$$

1.b

$$\mu^*(X) = \sum_{x \in A} c(x),$$

so one is finite if and only if the other is.

1.c

Assume for each x we have $c(x) < \infty$. Then since A is countable we have that

$$\{X \setminus A\} \cup \{\{x\} \mid x \in A\}$$

is a countable partition of X. Since $\mu^*(X \setminus A) = 0$ and $\mu^*(\{x\}) = c(x) < \infty$ each element of this partition has finite outer measure and hence X is σ-finite.

Assume μ^* is σ-finite. Then there is a countable cover $(X_i)_{i \in \mathbb{N}}$ of X such that for each i we have $\mu^*(X_i) < \infty$. Let $x \in A$. Then there is an i with $x \in X_i$. Now

$$c(x) = \mu^*(\{x\}) \leq \mu^*(X_i) < \infty.$$

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By the measurability of A we have

$$\begin{aligned}\mu^*(B) &= \mu^*(B \setminus A) + \mu^*(B \cap A) \\ &= \mu^*(B \setminus A) + \mu^*(A).\end{aligned}$$

If $\mu(A) < \infty$ then we can substract it from both sides and obtain the equation we have to prove. If $\mu^*(A) = \infty$ then by monotonicity we also have $\mu^*(B) = \infty$. Therefore in that case the right hand side of the desired equation is of the form "$\infty - \infty$", which is undefined.

Note that we did not assume A to be measurable to prove the equation. The equation does not hold for all unmeasurable B but for some. Example: Consider $X = \{0, 1\}$ and μ^* be the outer measure with $\mu^*(\{0\}) := \mu^*(\{1\}) := \mu^*(\{0, 1\} := 1$. Then the only measurable sets are $\emptyset, X$ i.e. $A := B := \{0\}$ is not measurable. Still

$$\mu^*(B \setminus A) = \mu^*(\emptyset) = 0 = 1 - 1 = \mu^*(B) - \mu^*(A).$$

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3.a

We have the disjoint unions

$$A \cup B = A \cup (B \setminus A)$$

and

$$B = (B \cap A) \cup (B \setminus A).$$

Thus

$$\begin{aligned}\mu^*(A \cup B) &= \mu(A) + \mu(B \setminus A) \\ &= \mu(A) + \mu(B) - \mu(A \cap B)\end{aligned}$$

3.b

From a) we obtain

$$\begin{aligned}\mu^*(A \cup B \cup C) &= \mu^*(A \cup B) + \mu(C) - \mu^*((A \cup B) \cap C) && (1) \\ &= \mu^*(A) + \mu(B) - \mu(A \cap B) + \mu(C) - \mu^*((A \cup B) \cap C) && (2)\end{aligned}$$

and

$$\begin{aligned}\mu^*((A \cup B) \cap C) &= \mu^*((A \cap C) \cup (B \cap C)) \\ &= \mu^*(A \cap C) + \mu^*(B \cap C) - \mu^*((A \cap C) \cap (B \cap C)) \\ &= \mu^*(A \cap C) + \mu^*(B \cap C) - \mu^*(A \cap B \cap C). && (3)\end{aligned}$$

Plugging Equation (3) into Equation (2) we obtain the desired result.

3.c

We prove

$$\mu(\bigcup_{i=1}^{k} A_i) = \sum_{s \subset \{1,\ldots,k\}, |s| \geq 1} (-1)^{|s|-1} \mu(\bigcap_{i \in s} A_i) \qquad (4)$$

by induction on k.

For $k=0$ both sides in Equation (4) are zero. Hence assume that Equation (4) holds for k and any collection of sets. We have to show it for $k+1$.

$$\mu(\bigcup_{i=1}^{k+1} A_i) = \mu([\bigcup_{i=1}^{k} A_i] \cup A_{k+1})$$

and by a)

$$= \mu([\bigcup_{i=1}^{k} A_i]) + \mu(A_{k+1}) - \mu([\bigcup_{i=1}^{k} A_i] \cap A_{k+1}) \tag{5}$$

By induction hypotheses we have

$$\mu(\bigcup_{i=1}^{k} A_i) = \sum_{s\subset\{1,\dots,k\},|s|\geq 1} (-1)^{|s|-1}\mu(\bigcap_{i\in s} A_i) \tag{6}$$

and

$$\begin{aligned}
\mu([\bigcup_{i=1}^{k} A_i] \cap A_{k+1}) &= \mu(\bigcup_{i=1}^{k}[A_i \cap A_{k+1}]) \\
&= \sum_{s\subset\{1,\dots,k\},|s|\geq 1} (-1)^{|s|-1}\mu(\bigcap_{i\in s}[A_i \cap A_{k+1}) \\
&= -\sum_{s\subset\{1,\dots,k\},|s|\geq 1} (-1)^{|s|+1-1}\mu(\bigcap_{i\in s\cup\{k+1\}} A_i)
\end{aligned} \tag{7}$$

Plugging Equations (6) and (7) into Equation (5) we obtain

$$\begin{aligned}
\mu(\bigcup_{i=1}^{k+1} A_i) &= \sum_{s\subset\{1,\dots,k\},|s|\geq 1} (-1)^{|s|-1}\mu(\bigcap_{i\in s} A_i) + \mu(A_{k+1}) + \sum_{s\subset\{1,\dots,k\},|s|\geq 1} (-1)^{|s|+1-1}\mu(\bigcap_{i\in s\cup\{k+1\}} A_i) \\
&= \sum_{s\subset\{1,\dots,k+1\},|s|\geq 1} (-1)^{|s|-1}\mu(\bigcap_{i\in s} A_i)
\end{aligned}$$

4

4.a

Define λ according to the hint, i.e. for every set $A \in \mathcal{M}$ set

$$\lambda(A) = \sup\{\mu(B) - \nu(B) : B \subset A,\ \nu(B) < \infty, B \in \mathcal{M}\}.$$

Note that for every A we have $\emptyset \subset A,\ \nu(\emptyset) < \infty,\ \emptyset \in\subset \mathcal{M}$ so that

$$\lambda(A) \geq \mu(\emptyset) - \nu(\emptyset) = 0.$$

First we show that for all $A \in \mathcal{M}$ we have

$$\mu(A) = \nu(A) + \lambda(A).$$

If $\nu(A) = \infty$ then also $\mu(A) = \infty$ and the statement is clear. Hence it remains to consider $\nu(A) < \infty$. In this case it follows from the following Claim 1

Claim 1. Let $A \in \mathcal{M}$ with $\nu(A) < \infty$. Then

$$\lambda(A) = \mu(A) - \nu(A).$$

Proof. Let $B \subset A$. Then by additivity we have

$$\mu(A) = \mu(B) + \mu(A \setminus B),$$
$$\nu(A) = \nu(B) + \nu(A \setminus B).$$

Therefore

$$\begin{aligned}\mu(A) - \nu(A) &= \mu(B) - \nu(B) + \mu(A \setminus B) - \nu(A \setminus B) \\ &\geq \mu(B) - \nu(B).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda(A) &= \sup\{\mu(B) - \nu(B) : B \subset A,\ \nu(B) < \infty\} \\ &= \mu(A) - \nu(A).\end{aligned}$$

□

Now we check that λ is a measure.

- $\lambda(\emptyset) = 0$

$$\lambda(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0$$

- Additivity Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of disjoint sets and abbreviate

$$A = \bigcup_{i\in\mathbb{N}} A_i.$$

Let $(B^k)_{k\in\mathbb{N}}$ with $B^k \subset A,\ \nu(B^k) < \infty$ and

$$\lambda(A) = \lim_{k\to\infty} (\mu - \nu)(B^k).$$

Then for each k we have that

$$(\mu - \nu)(B^k) = (\mu - \nu)(\bigcup_{i\in\mathbb{N}} (B^k \cap A_i))$$

and by the additivity of μ, ν and $\nu(B^k) < \infty$ we have

$$\begin{aligned}&= \sum_{i\in\mathbb{N}} (\mu - \nu)(B^k \cap A_i) \\ &\leq \sum_{i\in\mathbb{N}} \lambda(A_i)\end{aligned}$$

We can conclude

$$\lambda(A) \leq \sum_{i\in\mathbb{N}} \lambda(A_i).$$

For each A_i let B_i^k with $\nu(B_i^k) < \infty$ and

$$\lambda(A_i) = \lim_{k\to\infty} (\mu - \nu)(A_i).$$

Then

$$\sum_{i\in\mathbb{N}} \lambda(A_i) = \lim_{N\to\infty} \sum_{i=1}^{N} \lambda(A_i) = \lim_{N\to\infty} \sum_{i=1}^{N} \lim_{k\to\infty} (\mu - \nu)(B_i^k) = \lim_{N\to\infty} \lim_{k\to\infty} \sum_{i=1}^{N} (\mu - \nu)(B_i^k)$$

and by additivity of μ, ν and $\nu(B_i^k) < \infty$ we get

$$\begin{aligned}\sum_{i=1}^{N} (\mu - \nu)(B_i^k) &= (\mu - \nu)(\bigcup_{i=1}^{N} B_i^k) \\ &\leq \lambda(A).\end{aligned}$$

From this we get

$$\sum_{i\in\mathbb{N}} \lambda(A_i) \leq \lambda(A),$$

and we can conclude

$$\lambda(A) = \sum_{i\in\mathbb{N}} \lambda(A_i),$$

proving the additivity of λ, finnishing the proof that λ is a measure.

4.b

Let $\Omega = \{0\}$ and $\mu(\Omega) = \nu(\Omega) = \infty$. Then any measure λ on Ω does the job.

4.c

Let $(X_i)_i$ be an increasing sequence of sets with $\nu(X_i) < \infty$ and $\bigcup_{i\in\mathbb{N}} X_i = X$. Let $A \in \mathcal{M}$. Then since λ is a measure we have

$$\lambda(A) = \lim_{i\to\infty} \lambda(A \cap X_i)$$

and by Claim 1 we have

$$= \lim_{i\to\infty} (\mu - \nu)(A \cap X_i),$$

which defines $\lambda(A)$ uniquely.

5

Define

$$\tilde{A} := \bigcap_{n\in\mathbb{N}} A_{\frac{1}{n}}.$$

Then clearly $A \subset \tilde{A}$. The converse also holds: Let $x \in \tilde{A}$ and $0 < r \leq 1$. Then there is an n with $\frac{1}{n} \leq r$. For that n we have

$$x \in A \subset A_{\frac{1}{n}} \subset A_r.$$

Therefore $x \in A$ and we can conclude

$$A = \tilde{A}.$$

$\tilde{A}$ is measurable since it is an intersection of countably many measurable sets. Since for every n we have $\mu^*(A_{\frac{1}{n}}) \leq \mu^*(A_1) < \infty$ we know that

$$\mu^*(\tilde{A}) = \lim_{n \to \infty} \mu^*(A_{\frac{1}{n}}).$$

The function $r \mapsto \mu^*(A_r)$ is increasing and bounded from below by 0. Hence the limit $r \to 0$ exists, call it c. By definition for all ε there is a δ such that for all $0 < r \leq \delta$ we have $|c - \mu^*(A_r)| \leq \varepsilon$. Hence for all $n \geq \frac{1}{\delta}$ we have $|c - \mu^*(A_{\frac{1}{n}})| \leq \varepsilon$. Thusthe limits coincide so that we can conclude

$$\lim_{r \to 0} \mu^*(A_r) = c = \lim_{n \to \infty} \mu^*(A_{\frac{1}{n}}) = \mu^*(\tilde{A}) = \mu^*(A).$$

6

Let Σ be the set of σ-algebras $\mathcal{G}$ that contain $\mathcal{F}$. Define

$$\mathcal{F}_0 := \bigcap \Sigma = \bigcap_{\mathcal{G} \in \Sigma} \mathcal{G}.$$

Clearly $\mathcal{F} \subset \mathcal{F}_0$. Since 2^X is a σ-algebra that contains $\mathcal{F}$, this intersection is nonempty. If we show that $\mathcal{F}_0$ is a σ-algebra, then that is the smallest σ-algebra, because it is contained in all σ-algebras that contain $\mathcal{F}$.

- $\emptyset \in \mathcal{F}_0$ Since each σ-algebra $\mathcal{G} \in \Sigma$ contains $\emptyset$, so does $\mathcal{F}_0$
- Closure unter countable unions Let $(A_i)_i$ be a sequence of sets with $A_i \in \mathcal{F}_0$. Then for each $\mathcal{G} \in \Sigma$ we have $A_i \in \mathcal{G}$. Since $\mathcal{G}$ is a σ-algebra, also $\bigcup_i A_i \in \mathcal{G}$. Hence $\bigcup_i A_i \in \mathcal{G}$.
- Closure under complements Let $A \in \mathcal{F}_0$. Then for each $\mathcal{G} \in \Sigma$ we have $A \in \mathcal{G}$. Since $\mathcal{G}$ is a σ-algebra, also $A^{\complement} \in \mathcal{G}$. Hence $A^{\complement} \in \mathcal{F}_0$.

Note, that we did not use that the $\mathcal{G} \in \Sigma$ satisfy $\mathcal{F} \subset \mathcal{G}$ in showing that $\mathcal{F}_0$ is a σ-algebra, so we have actually proven that any intersection of σ-algebras (over the same space) is a σ-algebra.