Aalto University School of Science
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MS-E1280 Measure and integral, fall 2019

Homework assignment 4
Topics: Sets of measure zero.

Deadline 2.12.2019 at 16:00.

1. Let $A \subset \mathbb{R}^n$. Show that $m^*(A) = 0$ if and only if for every $\varepsilon > 0$ there exist balls $B(x_i, r_i)$, $i = 1, 2, \dots$, such that
$$A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \quad \text{and} \quad \sum_{i=1}^{\infty} m^*(B(x_i, r_i)) < \varepsilon.$$

2. Let $\mathbb{Q}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in \mathbb{Q}, \, i = 1, \dots, n\}$ be the rational n-tuples in $\mathbb{R}^n$.
   (a) Show that for every $\varepsilon > 0$ there exist balls $B(x_i, r_i)$, $i = 1, 2, \dots$ such that
   $$\mathbb{Q}^n \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \quad \text{and} \quad \sum_{i=1}^{\infty} m^*(B(x_i, r_i)) < \varepsilon.$$
   (b) Show that $\bigcup_{i=1}^{\infty} B(x_i, r_i)$ is dense in $\mathbb{R}^n$.

3. Assume that $A \subset \mathbb{R}^n$ is a Lebesgue measurable set with $m^*(A) > 0$. Let $\mathbb{Q}^n$ be the rational n-tuples in $\mathbb{R}^n$ and let $B = \bigcup_{q \in \mathbb{Q}^n} (q + A)$. Show that $m^*(\mathbb{R}^n \setminus B) = 0$.

4. Assume that $f, \, g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Show that if $f(x) = g(x)$ for Lebesgue almost every $x \in \mathbb{R}^n$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

5. Let $f_i : \mathbb{R} \to \mathbb{R}$, $f_i(x) = x/i$, $i = 1, 2, \dots$.
   (a) Show that $f_i(x) \to 0$ for every $x \in \mathbb{R}$ as $i \to \infty$.
   (b) Does the sequence (f_i) converge almost uniformly in $\mathbb{R}$?
   (c) Does the sequence (f_i) converge in Lebesgue measure in $\mathbb{R}$?

6. Assume that f_i, $i = 1, 2, \ldots$, and f are μ-measurable functions on X.

(a) Show that

$$\{x \in X : \lim_{i\to\infty} f_i(x) \neq f(x)\} = \bigcup_{k=1}^{\infty}\bigcap_{j=1}^{\infty}\left\{x \in X : \sup_{i\geq j}|f_i(x) - f(x)| \geq \frac{1}{k}\right\}.$$

(b) If $f_i \to f$ μ-almost everywhere, then

$$\mu\left(\bigcap_{j=1}^{\infty}\left\{x \in X : \sup_{i\geq j}|f_i(x) - f(x)| \geq \varepsilon\right\}\right) = 0 \quad \text{for every} \quad \varepsilon > 0.$$