Solutions to Assignment 2

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1

$\underline{\mu^*(\emptyset) = 0}$ ✓

$\underline{\text{monotonicity}}$ ✓

$\underline{\sigma\text{-subadditivity}}$ Let $A_1, A_2, \ldots \subset X$ and $A = \bigcup_{i \in \mathbb{N}} A_i$. If there is an i with $\mu^*(B_i) = \infty$ then both sides in the inequality for σ-subadditivity are infinite by monotonicity. So assume that for all i we have $\mu^*(B_i) < \infty$. Let $\varepsilon > 0$. Then for each i there is a cover $B_i^1, B_i^2, \ldots$ of A_i with

$$\mu^*(A_i) \geq \sum_{j=1}^{\infty} \rho(B_j^i) - \varepsilon 2^{-i}.$$

Then $\{B_i^j \mid i, j \in \mathbb{N}\}$ is a countable cover of A and

$$\begin{aligned} \mu^*(A) &\leq \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \rho(B_i^j) \\ &\leq \sum_{i \in \mathbb{N}} (\mu^*(A_i) + \varepsilon 2^{-i}) \\ &= (\sum_{i \in \mathbb{N}} (\mu^*(A_i)) + 2\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ finishes the proof of σ-subadditivity.

2

Since for every $A \in \mathcal{F}$, $\{A\}$ is a cover of A we have $\mu^*(A) \leq \rho(A)$.

For the other direction, let $A \in \mathcal{F}$ and $B_1, B_2, \ldots$ be a cover of A. Then by σ-subadditivity of ρ we have

$$\sum_{i=1}^{\infty} \rho(B_i) \geq \rho(A).$$

Since the infimum of the left hand side over all such covers $B_1, B_2, \ldots$ yields $\mu^*(A)$, we get $\mu^*(A) \geq \rho(A)$.

3

3.a

We already know that μ^* is monotone. Hence each $B \supset A$ satisfies $\mu^*(B) \geq \mu^*(A)$, so that

$$\mu^*(A) \leq \inf\{\mu^*(B) : B \in \mathcal{F},\ A \subset B\}.$$

If $\mu^*(A) = \infty$ then the reverse inequality immediately holds true as well. If $\mu^*(A) < \infty$, let $A_1, A_2, \ldots$ be a cover of A with

$$\mu^*(A) \geq \sum_{i \in \mathbb{N}} \rho(A_i) - \varepsilon.$$

Let $B = \bigcup_{i \in \mathbb{N}} A_i$. Then

$$\mu^*(B) \leq \rho(B) \leq \sum_{i \in \mathbb{N}} \rho(A_i) \leq \mu^*(A) + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain

$$\mu^*(A) \geq \inf\{\mu^*(B) : B \in \mathcal{F},\ A \subset B\}.$$

3.b

Let $B \in \mathcal{F}$ and $A \subset X$. It suffices to show that

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \setminus B).$$

If $\mu^*(A)$ is infinite then we are done. So assume $\mu^*(A)$ is finite, and let $B_1, B_2, \ldots$ a cover of A with

$$\mu^*(A) \geq \sum_{i \in \mathbb{N}} \rho(B_1) - \varepsilon.$$

Then $B_1 \cap B,\ B_2 \cap B, \ldots$ and $B_1 \setminus B, B_2 \setminus B, \ldots$ are covers of $A \cap B$ and $A \setminus B$ respectively with their elements in $\mathcal{F}$ and we have by σ-additivity of ρ that

$$\begin{aligned} \sum_{i \in \mathbb{N}} \rho(B_1) &= \sum_{i \in \mathbb{N}} \rho(B_1 \cap B) + \sum_{i \in \mathbb{N}} \rho(B_1 \setminus B) \\ &\geq \mu^*(A \cap B_1) + \mu^*(A \setminus B_1), \end{aligned}$$

so that we can conclude

$$\mu^*(A) \geq \mu^*(A \cap B_1) + \mu^*(A \setminus B_1) - \varepsilon,$$

and letting $\varepsilon \to 0$ finishes the proof.

3.c

From a) take a sequence $B_1, B_2, \ldots \in \mathcal{F}$ such that for ever n we have $A \subset B_n$ and

$$\mu^*(A) = \lim_{n \to \infty} \mu^*(B_n).$$

Define $B := \bigcap_{n\in\mathbb{N}} B_n$. Then $B \subset A$ and for every i we have

$$\mu^*(B) \leq \mu^*(B_i)$$

which implies

$$\mu^*(B) = \mu^*(A).$$

Since $\mathcal{F}$ is a σ-algebra we have $B \in \mathcal{F}$ and since all sets in $\mathcal{F}$ are measurable so is B.

4

4.a

Recall that for each A we have

$$\mu^*(A) = \sup_{\delta>0} \mu^*_\delta(A).$$

By 1) we already know that each μ^*_δ is an outer measure. Then $\mu^*_\delta(\emptyset) = 0$ and monotonicity directly carry over for the supremum. For σ-subadditivity let $A_1, A_2, \ldots \subset \mathbb{R}^n$. Then

$$\begin{aligned}
\mu^*(\bigcup_i A_i) &= \sup_{\delta>0} \mu^*_\delta(\bigcup_i A_i) \\
&\leq \sup_{\delta>0} \sum_i \mu^*_\delta(A_i) \\
&\leq \sum_i \sup_{\delta>0} \mu^*_\delta(A_i) \\
&= \sum_i \mu^*(A_i)
\end{aligned}$$

4.b

According to Theorem 1.45 is suffices to show that for any two sets $A, B \subset \mathbb{R}^n$ with $\operatorname{dist}(A, B) > 0$ we have

$$\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B);$$

the reverse inequality follows from subadditivity.

So let $A, B \subset \mathbb{R}^n$ and denote $\delta_0 := \operatorname{dist}(A, B) > 0$. Let $\varepsilon > 0$. Take $\delta \in (0, \delta_0/2)$ such that the μ^*_δ-measures of $A, B, A \cup B$ differs by their μ^*-measures by at most ε. Define $C := A \cup B$. Let $C_1, C_2, \ldots$ be a cover of C with diameters at most δ and

$$\mu^*_\delta(C) \geq \sum_{i\in\mathbb{N}} \rho(C_i) - \varepsilon.$$

That means for no i does C_i intersect both A and B. That means

$$\{C_i : i \in \mathbb{N},\ C_i \cap A \neq \emptyset\},$$
$$\{C_i : i \in \mathbb{N},\ C_i \cap B \neq \emptyset\}$$

are disjoint covers of A and B respectively, so that

$$\begin{aligned}\mu_\delta^*(C) &\geq \sum_{i\in\mathbb{N}} \rho(C_i) - \varepsilon \\ &\geq \sum_{i\in\mathbb{N},\ C_i\cap A\neq\emptyset} \rho(C_i) + \sum_{i\in\mathbb{N},\ C_i\cap B\neq\emptyset} \rho(C_i) - \varepsilon \\ &\geq \mu_\delta^*(A) + \mu^* v_\delta(B) - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ finishes the proof.

4.c

Let $A \subset \mathbb{R}^n$. For each n let $A_1^n, A_2^n, \ldots \mathcal{F}$ be a cover of A with diameters at most $1/n$ and

$$\mu_{1/n}^*(A) \geq \sum_{i\in\mathbb{N}} \rho(A_i^n) - 1/n.$$

Set

$$B := \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_i^n.$$

Then for each n

$$B \subset \bigcup_{i\in\mathbb{N}} A_i^n$$

and $A \subset B$ and B is Borel and

$$\begin{aligned}\mu^*(B) &= \lim_{\delta\to 0} \mu_\delta^*(B) \\ &= \lim_{n\to\infty} \mu_{1/n}^*(B) \\ &\leq \lim_{n\to\infty} \sum_{i\in\mathbb{N}} \rho(A_i^n) \\ &\leq \lim_{n\to\infty} \mu_{1/n}^*(A) - 1/n \\ &= \lim_{n\to\infty} \mu_{1/n}^*(A) \\ &= \mu^*(A).\end{aligned}$$

5

5.a

Let $E \subset \mathbb{R}^n$. By Corollary 1.44

$$\begin{aligned}\mu^*(E) &= \inf\{\mu^*(G) : E \subset G,\ G \text{ open}\} \\ &= \inf\{\nu^*(G) : E \subset G,\ G \text{ open}\} \\ &= \nu^*(E)\end{aligned}$$

5.b

Since compact sets are closed, for all compact K we have $\mu^*(K) = \nu^*(K)$. Then the result follows from (c).

5.c

By (a) it suffices to prove for all open G that $\mu^*(G) = \nu^*(G)$. So let G be open. Hence G is measurable. Then by Corollary 1.44

$$\begin{aligned}\mu^*(G) &= \sup\{\mu^*(K) : K \subset G,\ K \text{ compact}\} \\ &= \sup\{\nu^*(K) : K \subset G,\ K \text{ compact}\} \\ &= \nu^*(G)\end{aligned}$$