



**Aalto University**  
School of Electrical  
Engineering

# **ELEC-E8740 — Extended and Unscented Kalman Filtering**

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# Intended Learning Outcomes

After this lecture, you will be able to:

- recognize the challenges for filtering in nonlinear state-space models,
- describe and employ the extended and unscented Kalman filters for nonlinear state-space models

# Recap: Filtering and the Kalman Filter

- The filtering approach iterates between two steps:
  - 1 Prediction:  $\hat{\mathbf{x}}_{n-1|n-1}, \mathbf{P}_{n-1|n-1} \Rightarrow \hat{\mathbf{x}}_{n|n-1}, \mathbf{P}_{n|n-1}$
  - 2 Measurement update:  $\hat{\mathbf{x}}_{n|n-1}, \mathbf{P}_{n|n-1} \Rightarrow \hat{\mathbf{x}}_{n|n}, \mathbf{P}_{n|n}$
- The **Kalman filter** is the optimal filter for linear state-space models
  - 1 Prediction:

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{F}_n \hat{\mathbf{x}}_{n-1|n-1}$$

$$\mathbf{P}_{n|n-1} = \mathbf{F}_n \mathbf{P}_{n-1|n-1} \mathbf{F}_n^T + \mathbf{Q}_n$$

- 2 Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_n^T (\mathbf{G}_n \mathbf{P}_{n|n-1} \mathbf{G}_n + \mathbf{R}_n)^{-1}$$

$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathbf{G}_n \hat{\mathbf{x}}_{n|n-1})$$

$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_n \mathbf{P}_{n|n-1} \mathbf{G}_n + \mathbf{R}_n) \mathbf{K}_n^T$$

# Discrete-Time Nonlinear State-Space Model

- Discrete-time nonlinear state-space model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Process and measurement noises ( $\mathbf{q}_n$  and  $\mathbf{r}_n$ ):

$$E\{\mathbf{q}_n\} = \mathbf{0}, \text{Cov}\{\mathbf{q}_n\} = \mathbf{Q}_n$$

$$E\{\mathbf{r}_n\} = \mathbf{0}, \text{Cov}\{\mathbf{r}_n\} = \mathbf{R}_n$$

- Initial conditions:

$$E\{\mathbf{x}_0\} = \mathbf{m}_0, \text{Cov}\{\mathbf{x}_0\} = \mathbf{P}_0$$

# Filtering for Nonlinear Models

- For most nonlinear models, exact prediction and/or update steps can not be found
- Example: Prediction for general nonlinear model

$$\hat{\mathbf{x}}_{n|n-1} = E\{\mathbf{x}_n \mid \mathbf{y}_{1:n-1}\}$$

Approximations to the exact solutions are required!

## Linearized Model: Prediction (1/2)

- State estimate from  $t_{n-1}$ :  $\hat{\mathbf{x}}_{n-1|n-1}$ ,  $\mathbf{P}_{n-1|n-1}$
- Linearization around  $\hat{\mathbf{x}}_{n-1|n-1}$  (dynamic model):

$$\begin{aligned}\mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n \\ &\approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n\end{aligned}$$

- Predicted mean:

$$\begin{aligned}\hat{\mathbf{x}}_{n|n-1} &= E\{\mathbf{x}_n \mid \mathbf{y}_{1:n-1}\} \\ &\approx E\{\mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n \mid \mathbf{y}_{1:n-1}\} \\ &= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x E\{\mathbf{x}_{n-1} \mid \mathbf{y}_{1:n-1}\} - \mathbf{F}_x \hat{\mathbf{x}}_{n-1|n-1} \\ &= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x \hat{\mathbf{x}}_{n-1|n-1} - \mathbf{F}_x \hat{\mathbf{x}}_{n-1|n-1} \\ &= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})\end{aligned}$$

## Linearized Model: Prediction (2/2)

- State estimate from  $t_{n-1}$ :  $\hat{\mathbf{x}}_{n-1|n-1}$ ,  $\mathbf{P}_{n-1|n-1}$
- Linearization around  $\hat{\mathbf{x}}_{n-1|n-1}$  (dynamic model):

$$\mathbf{x}_n \approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n,$$

- Covariance:

$$\begin{aligned}\mathbf{P}_{n|n-1} &= \mathbb{E}\{(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T \mid \mathbf{y}_{1:n-1}\} \\ &\approx \mathbb{E}\{[\mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n - \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})] \\ &\quad \times [\dots]^T \mid \mathbf{y}_{1:n-1}\} \\ &= \mathbb{E}\{[\mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n][\dots]^T \mid \mathbf{y}_{1:n-1}\} \\ &= \mathbf{F}_x \mathbb{E}\{(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1})(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1})^T \mid \mathbf{y}_{1:n-1}\} \mathbf{F}_x^T \\ &\quad + \mathbb{E}\{\mathbf{q}_n \mathbf{q}_n^T \mid \mathbf{y}_{1:n-1}\} \\ &= \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^T + \mathbf{Q}_n\end{aligned}$$



# Linearized Model: Measurement Update (1/3)

- Prediction from  $t_{n-1}$  to  $t_n$ :  $\hat{\mathbf{x}}_{n|n-1}$ ,  $\mathbf{P}_{n|n-1}$
- Linearization around  $\hat{\mathbf{x}}_{n|n-1}$ :

$$\begin{aligned}\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n \\ &\approx \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{r}_n\end{aligned}$$

- Regularized linear least squares:

$$\begin{aligned}J_{\text{ReLS}}(\mathbf{x}_n) &= (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}))^T \mathbf{R}_n^{-1} \\ &\quad \times (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})) \\ &\quad + (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) \\ \hat{\mathbf{x}}_{n|n} &= \underset{\mathbf{x}_n}{\text{argmin}} J_{\text{ReLS}}(\mathbf{x}_n)\end{aligned}$$

## Linearized Model: Measurement Update (2/3)

- Regularized linear least squares:

$$\begin{aligned} J_{\text{ReLS}}(\mathbf{x}_n) &= (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}))^T \mathbf{R}_n^{-1} \\ &\quad \times (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) - \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})) \\ &\quad + (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) \end{aligned}$$

- Change of variables:  $\mathbf{z}_n = \mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}$ :

$$\begin{aligned} J_{\text{ReLS}}(\mathbf{x}_n) &= (\mathbf{z}_n - \mathbf{G}_x \mathbf{x}_n)^T \mathbf{R}_n^{-1} (\mathbf{z}_n - \mathbf{G}_x \mathbf{x}_n) \\ &\quad + (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T \mathbf{P}_{n|n-1}^{-1} (\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) \end{aligned}$$

- Solution (see Chapters 2.4, 5.2):

$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{z}_n - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1})$$

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_x^T (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^T + \mathbf{R}_n)^{-1}$$

$$\mathbf{P}_{n|n} \approx \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^T + \mathbf{R}_n) \mathbf{K}_n^T$$

# Linearized Model: Measurement Update (3/3)

- Measurement update:

$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{z}_n - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1})$$

- Substitution of  $\mathbf{z}_n = \mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}$ :

$$\begin{aligned}\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1} - \mathbf{G}_x \hat{\mathbf{x}}_{n|n-1}) \\ &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}))\end{aligned}$$

# Linearized Model: Summary

- Model approximation:

$$\mathbf{x}_n \approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n$$

$$\mathbf{y}_n \approx \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{r}_n$$

- Prediction:

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}),$$

$$\mathbf{P}_{n|n-1} = \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^T + \mathbf{Q}_n,$$

- Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_x^T (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^T + \mathbf{R}_n)^{-1},$$

$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1})),$$

$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^T + \mathbf{R}_n) \mathbf{K}_n^T.$$

# Extended Kalman Filter

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## Algorithm 1 Extended Kalman Filter

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- 1: Initialize  $\hat{\mathbf{x}}_{0|0} = \mathbf{m}_0$ ,  $\mathbf{P}_{0|0} = \mathbf{P}_0$
- 2: **for**  $n = 1, 2, \dots$  **do**
- 3:     Prediction (time update):

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1})$$
$$\mathbf{P}_{n|n-1} = \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^T + \mathbf{Q}_n$$

- 4:     Measurement update:

$$\mathbf{K}_n = \mathbf{P}_{n|n-1} \mathbf{G}_x^T (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x + \mathbf{R}_n)^{-1}$$
$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n (\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}))$$
$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x + \mathbf{R}_n) \mathbf{K}_n^T$$

- 5: **end for**
-

## Example: Object Tracking (1/3)

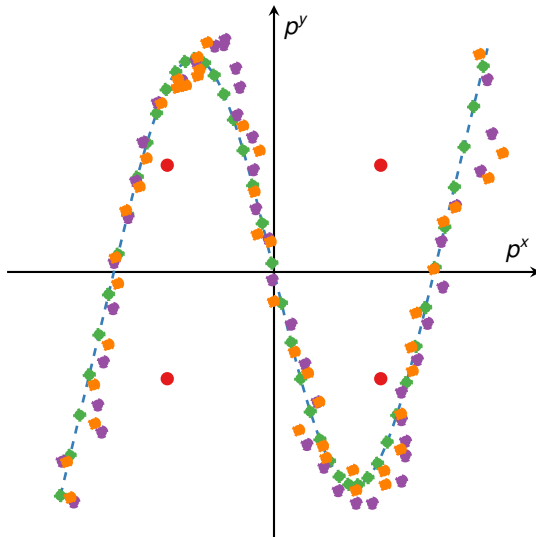
- Quasi-constant turn model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t) \cos(\varphi(t)) \\ v(t) \sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(t)$$

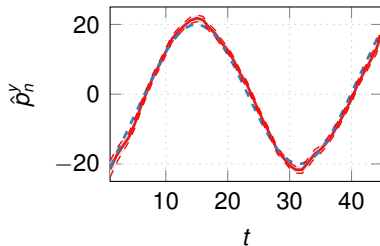
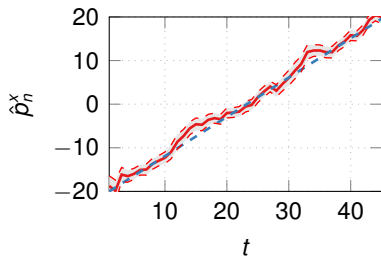
- Range (distance) measurements:

$$\mathbf{y}_n = \begin{bmatrix} |\mathbf{p}_n - \mathbf{p}_1^S| \\ |\mathbf{p}_n - \mathbf{p}_2^S| \\ \vdots \\ |\mathbf{p}_n - \mathbf{p}_K^S| \end{bmatrix} + \mathbf{r}_n$$

## Example: Object Tracking (2/3)



# Example: Object Tracking (3/3)



Position RMSE: 3.83 m

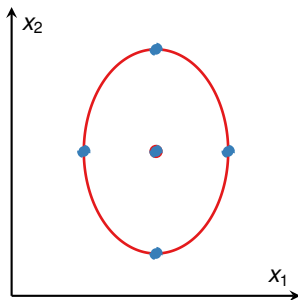


# Nonlinear Transformations of Random Variables

## (1/3)

- Given: Random variable  $\mathbf{x}$  with mean  $\mathbf{m}$  and covariance  $\mathbf{P}$
- Choose points  $\mathbf{x}^j$  and weights  $w_m^j, w_P^j$  such that:

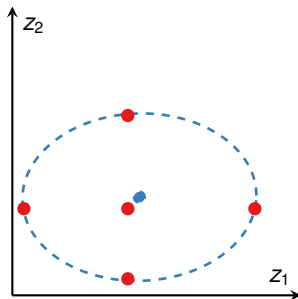
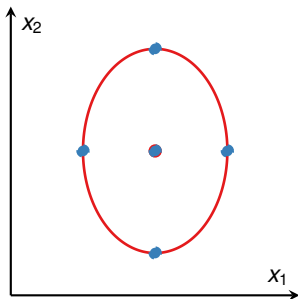
$$\mathbf{m} = \sum_{j=0}^{J-1} w_m^j \mathbf{x}^j, \quad \mathbf{P} = \sum_{j=0}^{J-1} w_P^j (\mathbf{x}^j - \mathbf{m})(\mathbf{x}^j - \mathbf{m})^T,$$



# Nonlinear Transformations of Random Variables (2/3)

- Given: Points  $\mathbf{x}^j$  and weights  $w_m^j, w_p^j$
- Nonlinear transformation:  $\mathbf{z} = \mathbf{h}(\mathbf{x})$
- Transformed points:

$$\mathbf{z}^j = \mathbf{h}(\mathbf{x}^j)$$



# Nonlinear Transformations of Random Variables

## (3/3)

- Given: Points  $\mathbf{x}^j$  and weights  $w_m^j, w_p^j$
- Nonlinear transformation:  $\mathbf{z} = \mathbf{h}(\mathbf{x})$
- Transformed points:

$$\mathbf{z}^j = \mathbf{h}(\mathbf{x}^j)$$

- Moments of the transformed variable

$$E\{\mathbf{z}\} \approx \sum_{j=1}^J w_m^j \mathbf{z}^j$$

$$\text{Cov}\{\mathbf{z}\} \approx \sum_{j=1}^J w_p^j (\mathbf{z}^j - E\{\mathbf{z}\})(\mathbf{z}^j - E\{\mathbf{z}\})^T$$

$$\text{Cov}\{\mathbf{x}, \mathbf{z}\} \approx \sum_{j=1}^J w_p^j (\mathbf{x}^j - \mathbf{m})(\mathbf{z}^j - E\{\mathbf{z}\})^T$$

# Unscented Transform

- **Unscented Transform:** One way of choosing  $\mathbf{x}^j$ ,  $w_m^j$  and  $w_P^j$ , uses  $2L + 1$  points
- Location of the sigma-points:

$$\mathbf{x}^0 = \mathbf{m}$$

$$\mathbf{x}^j = \mathbf{m} + \sqrt{L + \lambda} [\sqrt{\mathbf{P}}]_j, \quad j = 1, \dots, L$$

$$\mathbf{x}^j = \mathbf{m} - \sqrt{L + \lambda} [\sqrt{\mathbf{P}}]_{(j-L)}, \quad j = L + 1, \dots, 2L$$

- Weights of the sigma-points:

$$w_m^0 = \frac{\lambda}{L + \lambda}$$

$$w_P^0 = \frac{\lambda}{L + \lambda} + (1 - \alpha^2 + \beta)$$

$$w_m^j = w_P^j = \frac{1}{2(L + \lambda)}, \quad j = 1, \dots, 2L$$

# Unscented Transform: Prediction (1/2)

- Dynamic model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Sigma-points with  $\mathbf{m} = \hat{\mathbf{x}}_{n-1|n-1}$ ,  $\mathbf{P} = \mathbf{P}_{n-1|n-1}$ :

$$\mathbf{x}_{n-1}^0 = \hat{\mathbf{x}}_{n-1|n-1}$$

$$\mathbf{x}_{n-1}^j = \hat{\mathbf{x}}_{n-1|n-1} + \sqrt{L + \lambda} \left[ \sqrt{\mathbf{P}_{n-1|n-1}} \right]_j, \quad j = 1, \dots, L$$

$$\mathbf{x}_{n-1}^j = \hat{\mathbf{x}}_{n-1|n-1} - \sqrt{L + \lambda} \left[ \sqrt{\mathbf{P}_{n-1|n-1}} \right]_{(j-L)}, \quad j = L + 1, \dots, 2L$$

# Unscented Transform: Prediction (2/2)

- Dynamic model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Transformed points:

$$\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j), \quad j = 0, \dots, 2L$$

- Moments of the prediction:

$$\hat{\mathbf{x}}_{n|n-1} = \sum_{j=0}^{2L} w_m^j \mathbf{x}_n^j$$

$$\mathbf{P}_{n|n-1} = \sum_{j=0}^{2L} w_c^j (\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})(\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})^T + \mathbf{Q}_n$$

# Unscented Transform: Measurement Update (1/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Recall: Alternative form of measurement update:

$$\begin{aligned}\mathbf{K}_n &= \text{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \text{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}^{-1}, \\ \hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \text{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}), \\ \mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n \text{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \mathbf{K}_n^T.\end{aligned}$$

- We can calculate  $\text{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$ ,  $\text{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$ , and  $\text{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$  using the unscented transform
- Sigma-points based on  $\hat{\mathbf{x}}_{n|n-1}$ ,  $\mathbf{P}_{n|n-1}$ :

$$\mathbf{x}_n^0 = \hat{\mathbf{x}}_{n|n-1}$$

$$\mathbf{x}_n^j = \hat{\mathbf{x}}_{n|n-1} + \sqrt{L + \lambda} \left[ \sqrt{\mathbf{P}_{n|n-1}} \right]_j, \quad j = 1, \dots, L$$

$$\mathbf{x}_n^j = \hat{\mathbf{x}}_{n|n-1} - \sqrt{L + \lambda} \left[ \sqrt{\mathbf{P}_{n|n-1}} \right]_{(j-L)}, \quad j = L + 1, \dots, 2L$$

# Unscented Transform: Measurement Update (2/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- Transformed sigma-points:

$$\mathbf{y}_n^j = \mathbf{g}(\mathbf{x}_n^j), \quad j = 0, \dots, 2L$$

- Moments of the predicted  $\mathbf{y}_n$ :

$$E\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_m^j \mathbf{y}_n^j$$

$$\begin{aligned} \text{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} &= \sum_{j=0}^{2L} w_p^j (\mathbf{y}_n^j - E\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}) \\ &\quad \times (\mathbf{y}_n^j - E\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\})^T + \mathbf{R}_n \end{aligned}$$

$$\text{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_p^j (\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n-1})(\mathbf{y}_n^j - E\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\})^T$$

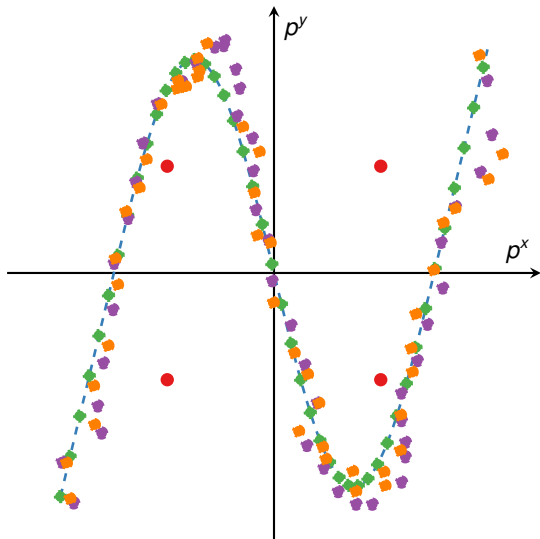


# Unscented Kalman Filter

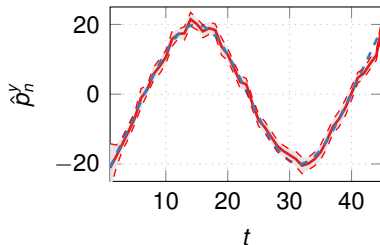
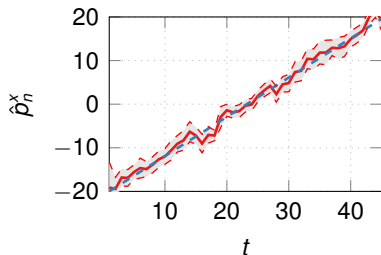
- Prediction:
  - Calculate the sigma-points using  $\hat{\mathbf{x}}_{n-1|n-1}$  and  $\mathbf{P}_{n-1|n-1}$
  - Propagate the sigma-points  $\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j)$
  - Calculate the mean and covariance  $\hat{\mathbf{x}}_{n|n-1}$ ,  $\mathbf{P}_{n|n-1}$
- Measurement update:
  - Calculate the sigma-points using  $\hat{\mathbf{x}}_{n|n-1}$  and  $\mathbf{P}_{n|n-1}$
  - Propagate the sigma-points  $\mathbf{y}_n^j = \mathbf{g}(\mathbf{x}_n^j)$
  - Calculate the mean and covariance  $E\{\mathbf{y}_n | \mathbf{y}_{1:n-1}\}$ ,  
 $\text{Cov}\{\mathbf{y}_n | \mathbf{y}_{1:n-1}\}$ ,  $\text{Cov}\{\mathbf{x}_n, \mathbf{y}_n | \mathbf{y}_{1:n-1}\}$
  - Perform the Kalman filter measurement update:

$$\mathbf{K}_n = \text{Cov}\{\mathbf{x}_n, \mathbf{y}_n | \mathbf{y}_{1:n-1}\} \text{Cov}\{\mathbf{y}_n | \mathbf{y}_{1:n-1}\}^{-1},$$
$$\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - E\{\mathbf{y}_n | \mathbf{y}_{1:n-1}\}),$$
$$\mathbf{P}_{n|n} = \mathbf{P}_{n|n-1} - \mathbf{K}_n \text{Cov}\{\mathbf{y}_n | \mathbf{y}_{1:n-1}\} \mathbf{K}_n^T.$$

# Example: Object Tracking (1/2)



## Example: Object Tracking (2/2)



Position RMSE: 1.45 m

# Unscented Transform: Choice of Parameters

- The parameter  $\lambda$  is actually:

$$\lambda = \alpha^2(L + \kappa) - L$$

- $\alpha$ ,  $\beta$ , and  $\kappa$  are tuning parameters
- $\kappa$  is usually set to 0
- $\alpha$  controls the spread of the sigma-points:

$$\sqrt{L + \lambda} = \sqrt{L + \alpha^2(L + \kappa) - L} = \alpha\sqrt{L}.$$

- Suggestions vary, e.g.,  $\alpha = 1 \times 10^{-3}$
- $\beta$  only affects the covariance weight, a good starting point is  $\beta = 2$

# Summary

- Nonlinear state-space models require approximative solutions
- The extended Kalman filter uses a linearization of the dynamic and measurement models
- The unscented Kalman filter uses a set of deterministic sigma-points (samples) to calculate the means and covariances