Aalto University School of Science
Department of Mathematics and Systems Analysis Kinnunen

MS-E1280 Measure and integral, fall 2019

Homework assignment 5
Topics: Measurable functions.

Deadline 9.12.2019 at 16:00.

1. Let $f : X \to [-\infty, \infty]$ be a function. Show that f is a μ-measurable function if and only if $\{x \in X : f(x) > q\}$ is a μ-measurable set for every $q \in \mathbb{Q}$.

2. Show that $f : X \to [-\infty, \infty]$ is μ-measurable if and only if the truncation
$$f_i(x) = \begin{cases} i, & f(x) \geq i, \\ f(x), & -i < f(x) < i, \\ -i, & f(x) \leq -i, \end{cases}$$
is μ-measurable for every $i = 1, 2, \dots$.

3. Show that a homemorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ maps Borel sets to Borel sets. Hint: $\{A \subset \mathbb{R}^n : f(A) \text{ is a Borel set}\}$ is a σ-algebra containing open sets.

4. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if and only if there are continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots$, such that $f = \lim_{i\to\infty} f_i$ almost everywhere in $\mathbb{R}^n$.

5. Let $f_i : X \to [-\infty, \infty]$ be μ-measurable functions with $|f_i| < \infty$ and $|f| < \infty$ μ-almost everywhere in X for every $i = 1, 2, \dots$. Assume that $f_i \to f$ μ-almost everywhere in X as $i \to \infty$. Prove the following assertions.

   (a) f is μ-measurable.

   (b) f is unique up to a set of μ-measure zero, that is, if g is another function such that $f_i \to g$ μ-almost everywhere in X, then f and g coincide outside a set of μ-measure zero.

   (c) If $g_i = f_i$ μ-almost everywhere in X for every $i = 1, 2, \dots$ and $g = f$ μ-almost everywhere in X, then $g_i \to g$ μ-almost everywhere in X as $i \to \infty$.

6. Assume that $f_i \to f$ and $g_i \to g$ in measure. Show that $f_i + g_i \to f + g$ in measure.