Solutions to Assignment 3

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November 27, 2019

1

1.a

For an open interval J denote by $\bar{J}$ its closure. Then

$$m^*(A) := \inf\{\sum_{\in\mathbb{N}} \operatorname{vol}(\bar{J}_i) : A \subset \bigcup_{i\in\mathbb{N}} \bar{J}_i\}$$

where the infimum is taken over all coverings of A by countably many open intervals $J_1, J_2, \ldots$. Define

$$\mu^*(A) := \inf\{\sum_{\in\mathbb{N}} \operatorname{vol}(J_i) : A \subset \bigcup_{i\in\mathbb{N}} J_i\}$$

where the infimum is taken over all coverings of A by countably many open intervals $J_1, J_2, \ldots$. We have to show that for all sets $A \subset \mathbb{R}^n$ we have

$$m^*(A) = \mu^*(A).$$

Since $\operatorname{vol}(\bar{J}_i) = \operatorname{vol}(J_i)$ and $J_i \subset \bar{J}_i$ we immediately get

$$m^*(A) \leq \mu^*(A).$$

If $m^*(A) = \infty$ then the reverse inequality holds clearly. Otherwise the reverse inequality let $\bar{J}_1, \bar{J}_2, \ldots$ be a covering of A with

$$\sum_{i\in\mathbb{N}} \operatorname{vol}(J_i) \leq m^*(A) + \varepsilon.$$

Then for each i let I_i be an open interval that contains $\bar{J}_i$ and with

$$\operatorname{vol}(I_i) \leq \operatorname{vol}(J_i) + 2^{-i}\varepsilon.$$

Then $I_1, I_2, \ldots$ is a covering of A with open intervals and thus

$$\begin{aligned}\mu^*(A) &\leq \sum_{i\in\mathbb{N}} \operatorname{vol}(I_i) \\ &\leq \sum_{i\in\mathbb{N}} (\operatorname{vol}(J_i) + \varepsilon 2^{-i})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in\mathbb{N}} \operatorname{vol}(J_i) + 2\varepsilon \\
&\le m^*(A) + 3\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$ finishes the proof of the reverse inequlity.

1.b

We have to show that for each compact A and $\varepsilon > 0$ there is a finite covering of A of closed intervals $J_1, \dots, J_n$ with

$$\sum_{i=1}^{n} \operatorname{vol}(J_i) \le m^*(A) + \varepsilon.$$

According to (a) there is covering of A of open intervals $I_1, I_2, \dots$ with

$$\sum_{i=1}^{\infty} \operatorname{vol}(I_i) \le \mu^*(A) + \varepsilon = m^*(A) + \varepsilon.$$

Since A is compact this covering has a finite subcover $J_1, \dots, J_n$. Then $\bar{J}_1, \dots, \bar{J}_n$ is a finite cover of closed intervals of A with

$$\sum_{i=1}^{n} \operatorname{vol}(\bar{J}_i) \le m^*(A) + \varepsilon.$$

2

Let $A \subset \mathbb{R}^n$. If A is measurable then the definition implies that for any cube Q we have

$$m^*(Q) = m^*(Q \cap A) + m^*(Q \setminus A).$$

Assume that for any cube Q we have

$$m^*(Q) = m^*(Q \cap A) + m^*(Q \setminus A).$$

We have to show that for every set $E \subset \mathbb{R}^n$ we have

$$m^*(E) = m^*(E \cap A) + m^*(E \setminus A).$$

By subadditivity we have

$$m^*(E) \le m^*(E \cap A) + m^*(E \setminus A).$$

It remains to show the reverse inequality. Since m^* is Radon we have that

$$m^*(E) = \inf\{m^*(U) : E \subset U,\ U \text{ open}\}.$$

Let U be open with $E \subset U$. Then there is a countable set $\mathcal{Q}$ of disjoint dyadic cubes with

$$E = \bigcup \mathcal{Q} := \bigcup_{Q\in\mathcal{Q}} Q.$$

Since cubes are measurable we have

$$m^*(U) = \sum_{Q \in \mathcal{Q}} m^*(Q)$$

and by assumption of the outer measure

$$\begin{aligned} &= \sum_{Q \in \mathcal{Q}} m^*(Q \cap A) + m^*(Q \setminus A) \\ &= \sum_{Q \in \mathcal{Q}} m^*(Q \cap A) + \sum_{Q \in \mathcal{Q}} m^*(Q \setminus A) \end{aligned}$$

by σ-subaddivitity

$$\begin{aligned} &\geq m^*(U \cap A) + m^*(U \setminus A) \\ &\geq m^*(E \cap A) + m^*(E \setminus A). \end{aligned}$$

3

$\mathcal{G}_\delta$ sets are Borel since they are countable intersections of open sets, and $\mathcal{F}_\sigma$-sets are measurable since they are countable unions of closed sets. Hence they are Lebesgue measurable, and adding or removing a set of measure zero does not change that.

It remains to show that every Lebesgue measurable set A is of the form (a) and (b). By Theorem 1.42 there is a sequence $F_1, F_2, \ldots$ of closed sets and a sequence $G_1, G_2, \ldots$ of open sets with $F_i \subset A \subset G_i$ and

$$\begin{aligned} \lim_{i \to \infty} m^*(F_i) = m^*(A), \\ \lim_{i \to \infty} m^*(G_i) = m^*(A). \end{aligned}$$

Then $F := \bigcup_i F_i$ and $G := \bigcap_i G_i$ satisfy $F \subset A \subset G$ and $F_i \subset F,\ G \subset G_i$ so that

$$\begin{aligned} \lim_{i \to \infty} m^*(F_i) \leq m^*(F) \leq m^*(A), \\ \lim_{i \to \infty} m^*(G_i) \geq m^*(G) \geq m^*(A), \end{aligned}$$

which implies

$$m^*(F) = m^*(G) = m^*(A)$$

and

$$m^*(A \setminus F) = m^*(G \setminus A) = .0$$

Hence A is a union $F \cup (A \setminus F)$ of an $\mathcal{F}_\sigma$ set an a set of measure zero, and also a $\mathcal{G}_\delta$ set with a set of measure zero removed $G \setminus (G \setminus A)$.

4

4.a

Let $A \subset \mathbb{R}^n$. The interior of A,

$$\mathring{A} = \{x \in A : \operatorname{dist}(x, A^{\complement}) > 0\} = A \setminus \partial A$$

is open. Thus A is the union of the open set $\mathring{A}$ and the measure zero set $A \setminus \mathring{A} \subset \partial A$ and hence is Lebesgue measurable.

4.b

$\mathbb{Q}^n$ is countable thus measurable, but $\partial \mathbb{Q}^n = \mathbb{R}^n$ has infinite measure.

5

5.a

Note that we use (b) here, but we don't use (a) in (b).

Since $A = \delta^{-1}\delta A$ it suffices to show that δA is measurable if A is measurable. So let A be measurable and $E \subset \mathbb{R}^n$. Then by (b) we have

$$\begin{aligned}
m^*(E \cap \delta A) + m^*(E \setminus \delta A) &= \delta^n m^*([\delta^{-n} E] \cap \delta) + \delta^n m^*([\delta^{-n} E] \setminus A) \\
&= \delta^n [m^*([\delta^{-n} E] \cap A) + m^*([\delta^{-n} E] \setminus A)] \\
&= \delta^n m^*(\delta^{-n} E) \\
&= m^*(E).
\end{aligned}$$

5.b

Note that the set J is an interval with $\operatorname{vol}(J) = v$ if and only if the set δJ is an interval with $\operatorname{vol}(\delta J) = \delta^n v$. Therefore

$$\begin{aligned}
m^*(\delta A) &= \inf\{\sum_{i \in \mathbb{N}} \operatorname{vol}(J_i) : J_1, J_2, \ldots \text{ intervals with } \delta A \subset \bigcup_{i \in \mathbb{N}} J_i\} \\
&= \inf\{\sum_{i \in \mathbb{N}} \operatorname{vol}(J_i) : J_1, J_2, \ldots \text{ intervals with } A \subset \bigcup_{i \in \mathbb{N}} \delta^{-1} J_i\} \\
&= \inf\{\sum_{i \in \mathbb{N}} \delta^n \operatorname{vol}(\delta^{-1} J_i) : \delta^{-1} J_1, \delta^{-1} J_2, \ldots \text{ intervals with } A \subset \bigcup_{i \in \mathbb{N}} \delta^{-1} J_i\} \\
&= \delta^n \inf\{\sum_{i \in \mathbb{N}} \operatorname{vol}(\delta^{-1} J_i) : \delta^{-1} J_1, \delta^{-1} J_2, \ldots \text{ intervals with } A \subset \bigcup_{i \in \mathbb{N}} \delta^{-1} J_i\} \\
&= \delta^n m^*(A)
\end{aligned}$$

6

Consider the function

$$f(r) := m^*[A \cap B(0,r)].$$

Then

$$\begin{aligned} f(0) &= 0, \\ \lim_{r\to\infty} f(r) &= m^*(\bigcup_{r>0} A \cap B(0,r)) \\ &= m^*(A) = \infty. \end{aligned}$$

Furthermore f is continuous because for $\varepsilon > 0$ we have

$$\begin{aligned} |f(r+\varepsilon) - f(r)| &= |m^*[A \cap B(0, r+\varepsilon)] - m^*[A \cap B(0,r)]| \\ &= m^*([A \cap B(0, r+\varepsilon)] \setminus [A \cap B(0,r)]) \\ &\leq m^*(B(0, r+\varepsilon) \setminus B(0,r)) \\ &= m^*[B(0,1)][(r+\varepsilon)^d - r^d] \\ &\to 0 \qquad \text{for } \varepsilon \to 0. \end{aligned}$$

and similarly

$$\begin{aligned} |f(r) - f(r-\varepsilon)| &\leq m^*[B(0,1)][(r)^d - (r-\varepsilon)^d] \\ &\to 0 \qquad \text{for } \varepsilon \to 0. \end{aligned}$$

Therefore f attains all values in $[0, \infty)$.