Aalto University School of Science
Department of Mathematics and Systems Analysis Kinnunen

MS-E1280 Measure and integral, fall 2019

Homework assignment 6
Topics: Integral.

Deadline 16.12.2019 at 16:00.

1. Let $f \in L^1(\mathbb{R}^n)$ and $\delta \neq 0$. Show that

$$\int_{\mathbb{R}^n} f(\delta x)\, dx = |\delta|^{-n} \int_{\mathbb{R}^n} f(x)\, dx.$$

2. Assume that $f, g \in L^1(\mathbb{R}^n)$ are such that

$$\int_A f\, dx = \int_A g\, dx$$

for every measurable set $A \subset \mathbb{R}^n$. Show that $f = g$ almost everywhere in $\mathbb{R}^n$.

3. Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set with $m(A) < \infty$ and $f : A \to [-\infty, \infty]$ be a Lebesgue measurable function such that $|f| < \infty$ almost everywhere in A. For every $\varepsilon > 0$ construct a bounded measurable function $g : A \to \mathbb{R}$ such that

$$m(\{x \in A : f(x) \neq g(x)\}) < \varepsilon.$$

Hint: Study the sets $\{x \in X : |f(x)| > i\}$.

Does the claim hold without the assumption $m(A) < \infty$?

4. Assume that $w : \mathbb{R}^n \to [0, \infty]$ is a Lebesgue measurable function and define $\mu(A) = \int_A w\, dx$, where $A \subset \mathbb{R}^n$ is a Lebesgue measurable set. Show that

$$\int_{\mathbb{R}^n} f\, d\mu = \int_{\mathbb{R}^n} f w\, dx$$

for every Lebesgue measurable function $f : \mathbb{R}^n \to [0, \infty]$.

Hint: First prove the claim for a simple function. Approximate by simple functions and use convergence theorems.

Does the corresponding claim hold for other measures than the Lebesgue measure?

5. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measure spaces, assume that μ is a measure on $(X, \mathcal{F})$ and $f : X \to Y$ a $(\mathcal{F}, \mathcal{G})$-measurable mapping. Define the image measure $f_{\#}\mu$ on $(Y, \mathcal{G})$ by $f_{\#}\mu(A) = \mu(f^{-1}(A))$, $A \in \mathcal{G}$. Show that $f_{\#}\mu$ is a measure on $(Y, \mathcal{G})$

6. (Continues) Let $\varphi : Y \to [0, \infty)$ be a $\mathcal{G}$-measurable function. Show that

$$\int_X \varphi(f(x))\, d\mu(x) = \int_Y \varphi(y)\, df_{\#}\mu(y).$$

   Hint: First prove the claim for a simple function φ.