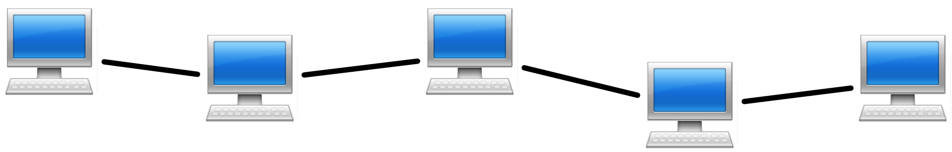
- **Weeks 1–2: informal introduction**
  - network = path 
- **Week 3: graph theory**
- **Weeks 4–7: models of computing**
  - what can be computed (efficiently)?
- **Weeks 8–11: lower bounds**
  - what cannot be computed (efficiently)?
- **Week 12: recap**

Week 10

– Ramsey theory

Avoiding cliques and independent sets

- **Can you construct graphs such that:**
  - there are N nodes
  - there is no clique of size n
  - there is no independent set of size n
- **For $n = 3$ and $N = 3, 4, 5, 6, \ldots$?**
  For $n = 4$ and $N = 4, 5, 6, 7, \ldots$?

Avoiding monochromatic sets

- **Can you construct *complete* graphs such that:**
  - there are N nodes
  - each edge coloured ***blue*** or ***orange***
  - there is no ***monochromatic*** set of size n
- **For $n = 3$ and $N = 3, 4, 5, 6, \ldots$?**
  For $n = 4$ and $N = 4, 5, 6, 7, \ldots$?

Monochromatic subsets

- Y = set with N elements, c colours, each k-subset of Y labelled with a colour
- X monochromatic: all k-subsets of X labelled with the same colour

Ramsey's theorem

- Y = set with N elements, c colours, each k-subset of Y labelled with a colour
- X **monochromatic**: all k-subsets of X labelled with the same colour
- For all c, k, n: if **N is large enough**, there is always a monochromatic subset of size n

Ramsey numbers

- Y = set with N elements, c colours, each k-subset of Y labelled with a colour
- X **monochromatic**: all k-subsets of X labelled with the same colour
- For all c, k, n: if $N \geq R_c(n; k)$, there is always a monochromatic subset of size n

Ramsey's theorem

- For all c, k, n there are numbers $R_c(n; k)$ s.t.: if we have $N \geq R_c(n; k)$ elements and we label each k-subset with one of c colours, there is a monochromatic subset of size n

Application

- We can show that $R_2(3; 2) = 6$
- Complete graph with 6 nodes, edges (= 2-subsets) labelled with 2 colours
- There is always a monochromatic subset of size 3

Application

- **We can show that $R_2(3; 2) = 6$**

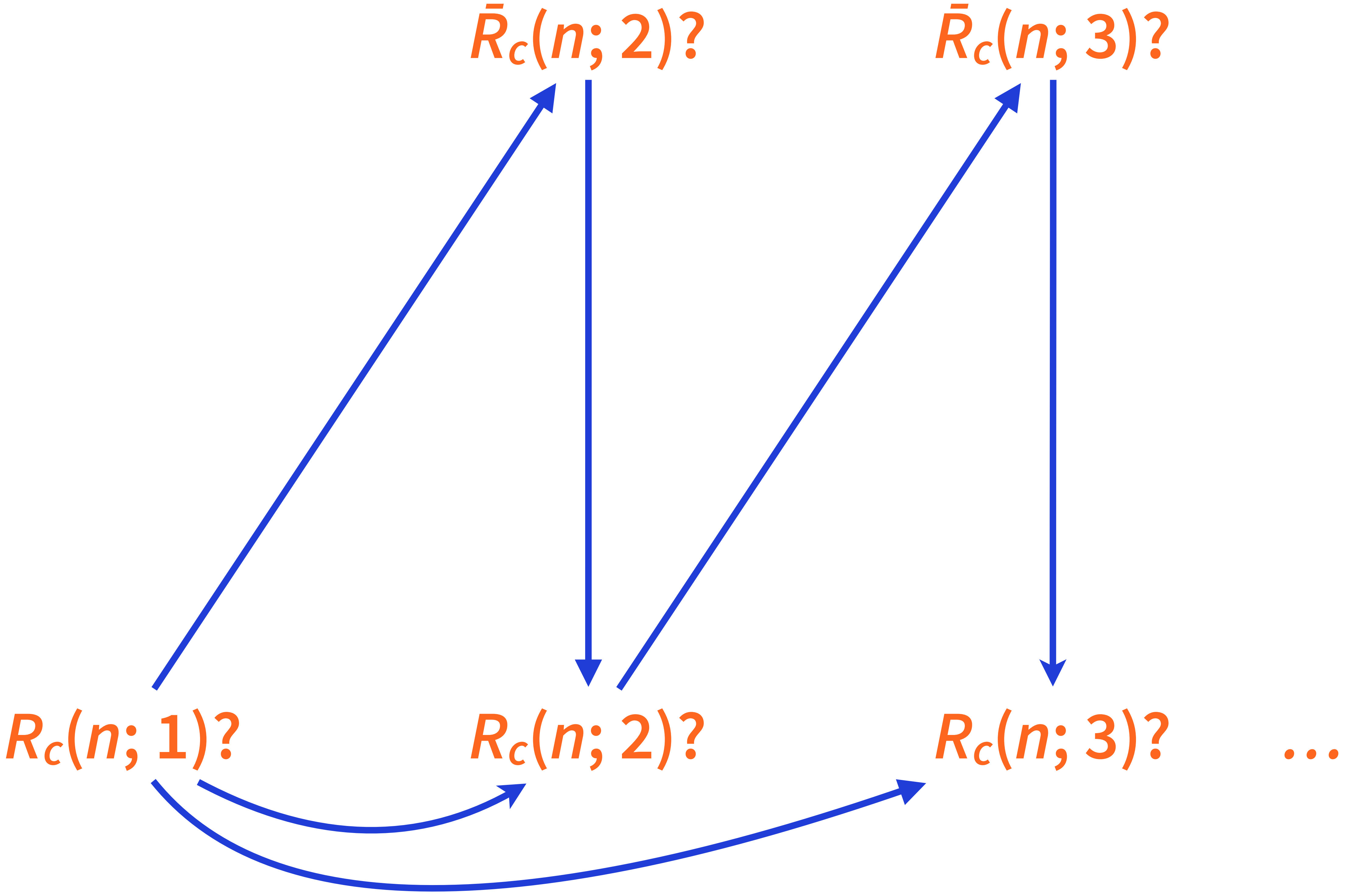
- **A graph with 6 nodes,**
  for each pair of nodes (= 2-subsets)
  edge may or may not exist (= 2 "colours")

- **There is always a clique or**
  an independent set of size 3

Ramsey's theorem

- **For all c, k, n there are numbers $R_c(n; k)$ s.t.: if we have $N \geq R_c(n; k)$ elements and we label each k-subset with one of c colours, there is a monochromatic subset of size n**
- **Proof…**

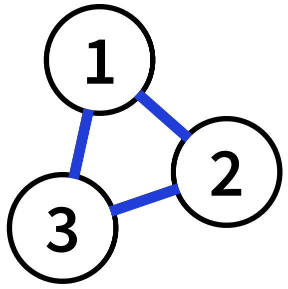
$R_c(n;1)$? $R_c(n;2)$? $R_c(n;3)$?   …



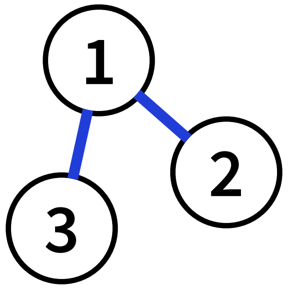
$\bar{R}_c(n;2)$?
$\bar{R}_c(n;3)$?
$R_c(n;1)$?
$R_c(n;2)$?
$R_c(n;3)$?
...

Almost monochromatic

- *Y* = set with *N* elements, *c* colours, each *k*-subset of *Y* labelled with a colour
- *X* monochromatic: all *k*-subsets of *X* labelled with the same colour

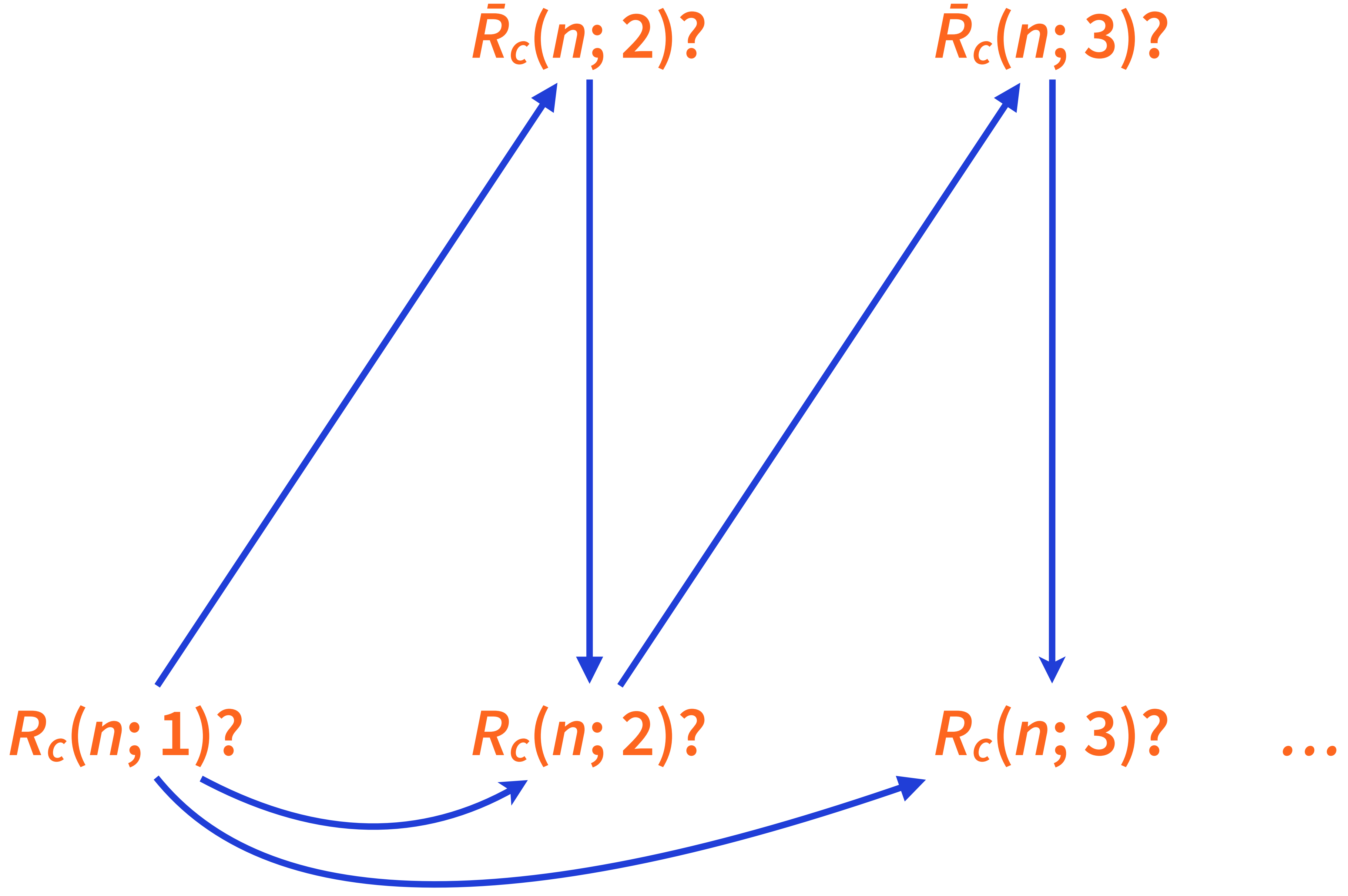


- *X* almost monochromatic: subsets with the *same minimum* have the same colour



Almost monochromatic

- **If we have $N \geq R_c(n; k)$ elements there is a *monochromatic* subset of size n**
- **If we have $N \geq \bar{R}_c(n; k)$ elements there is an *almost monochromatic* subset of size n**


$\bar{R}_c(n;2)$?
$\bar{R}_c(n;3)$?
$R_c(n;1)$?
$R_c(n;2)$?
$R_c(n;3)$?
...

$\bar{R}_c(n;2)$?

$\bar{R}_c(2;2)=2$

$M=\bar{R}_c(2;2)$
$\bar{R}_c(3;2)\leq 1+R_c(M;1)$

$M=\bar{R}_c(3;2)$
$\bar{R}_c(4;2)\leq 1+R_c(M;1)$

$\ldots$

$\bar{R}_c(n;3)$?

$\bar{R}_c(3;3)=3$

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$\ldots$

$R_c(n;1)$?

$R_c(n;1)\leq c\cdot(n-1)+1$

$R_c(n;2)$?

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$R_c(n;2)\leq \bar{R}_c(M;2)$

$R_c(n;3)$?

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$\bar{R}_c(n; 2)$?

$\bar{R}_c(2; 2) = 2$

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...

$R_c(n; 1)$?

$R_c(n; 1) \leq c \cdot (n-1)+1$

Lemma 10.3

$R_c(n; 2)$?

$M = R_c(n; 1)$
$R_c(n; 2) \leq \bar{R}_c(M; 2)$

$R_c(n; 3)$?

$M = R_c(n; 1)$
$R_c(n; 3) \leq \bar{R}_c(M; 3)$

$\bar{R}_c(n; 2)$?

$\bar{R}_c(n; 3)$?

$\bar{R}_c(2; 2) = 2$ **trivial** $\bar{R}_c(3; 3) = 3$

$M = \bar{R}_c(2; 2)$
$\bar{R}_c(3; 2) \leq 1 + R_c(M; 1)$

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...

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$\ldots$

Lemma 10.4

$R_c(n; 1)$?

$R_c(n; 1) \leq c \cdot (n-1) + 1$

$R_c(n; 2)$?

$M = R_c(n; 1)$
$R_c(n; 2) \leq \bar{R}_c(M; 2)$

$R_c(n; 3)$?

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...

$R_c(n; 1)$?

$R_c(n; 1) \leq c \cdot (n-1)+1$

Lemma 10.6

$R_c(n; 2)$?

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$R_c(n; 2) \leq \bar{R}_c(M; 2)$

$R_c(n; 3)$?

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$R_c(n; 1) \leq c \cdot (n-1)+1$

$R_c(n; 2)$?

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$R_c(n; 2) \leq \bar{R}_c(M; 2)$

$R_c(n; 3)$?

$M = R_c(n; 1)$
$R_c(n; 3) \leq \bar{R}_c(M; 3)$

- Pigeonhole principle
- $c \cdot (n-1)+1$ elements
- c-labelling of elements
- all labels have < n element: contradiction
- at least one label with n elements

$R_c(n; 1)$?

$R_c(n; 1) \leq c \cdot (n-1)+1$

Lemma 10.3

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$R_c(n; 3)$?

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$\bar{R}_c(n; 2)$?

$\bar{R}_c(2; 2) = 2$

trivial

$\bar{R}_c(n; 3)$?

$\bar{R}_c(3; 3) = 3$

- *k*-subsets are labelled
- need an (almost) monochromatic subset of size $k = n$
- any such set is (almost) monochromatic
- we only need $k = n$ elements

$\bar{R}_c(n; 2)$?

$\bar{R}_c(2; 2) = 2$

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Lemma 10.4

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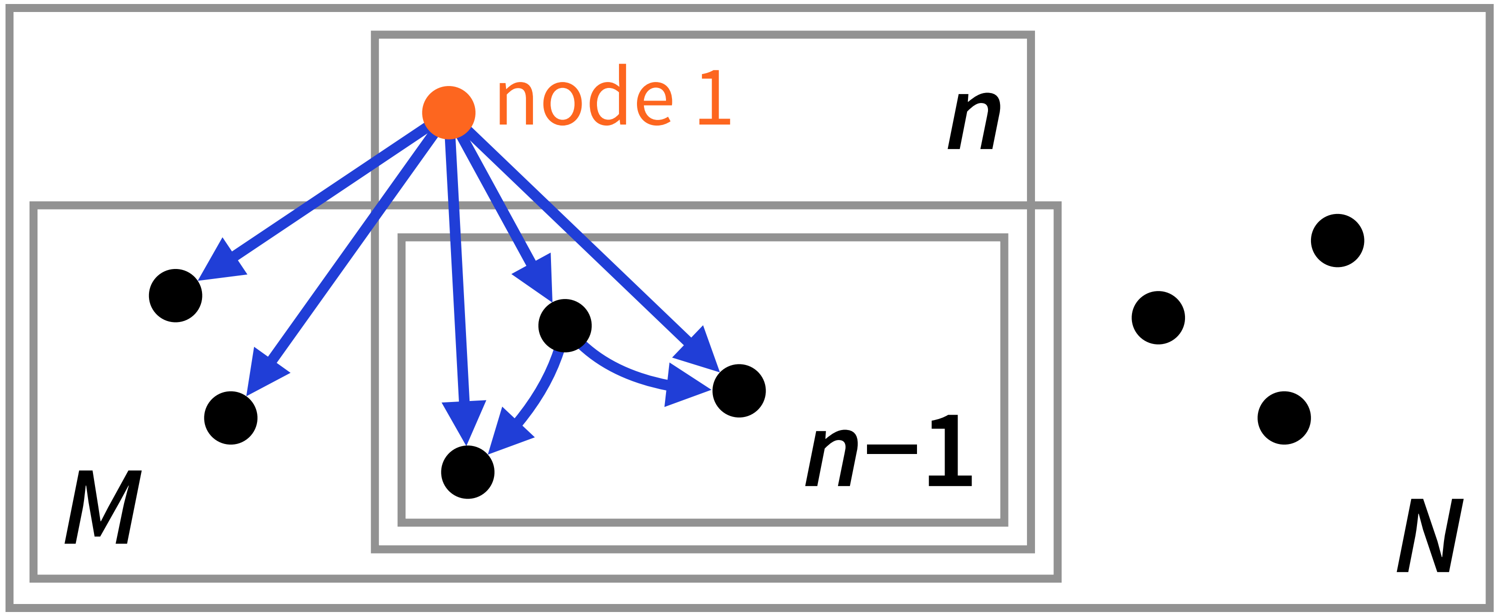

Lemma 10.4

$M = \bar{R}_c(n-1; k)$

$\bar{R}_c(n; k) \leq 1 + R_c(M; k-1) = N$

- ***M***: monochromatic for subsets containing 1
- ***n*−1**: almost monochromatic for other subsets
- ***n***: almost monochromatic for all subsets

$\bar{R}_c(n; 2)?$

$\bar{R}_c(2; 2) = 2$

$M = \bar{R}_c(2; 2)$
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$\bar{R}_c(n; 3)?$

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$R_c(n; 1)?$

$R_c(n; 1) \leq c \cdot (n-1)+1$

$R_c(n; 2)?$

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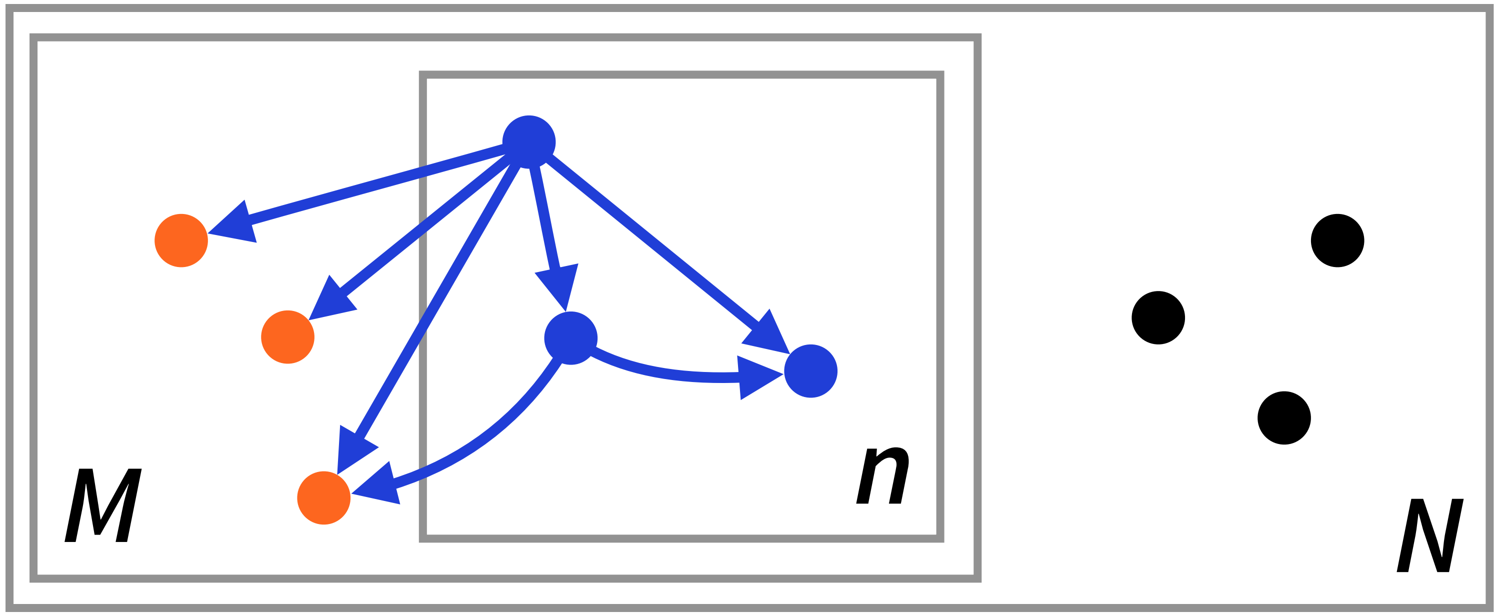

- ***M***: almost monochromatic
- colour of element i:
  common colour of subsets A with $\min(A) = i$
- ***n***: elements with same colour → monochromatic

Lemma 10.6

$M = R_c(n; 1)$

$R_c(n; k) \leq \bar{R}_c(M; k) = N$

$\bar{R}_c(n; 2)$?

$\bar{R}_c(2; 2) = 2$

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$R_c(n; 1)$?

$R_c(n; 1) \leq c \cdot (n-1)$

$R_c(n; 2)$?

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$R_c(n; 3)$?

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Summary

- **For all c, k, n there are numbers $R_c(n; k)$ s.t.: if we have $N \geq R_c(n; k)$ elements and we label each k-subset with one of c colours, there is a monochromatic subset of size n**
  - application for $k = 2$, $c = 2$: any graph with N nodes contains an independent set or a clique of size n

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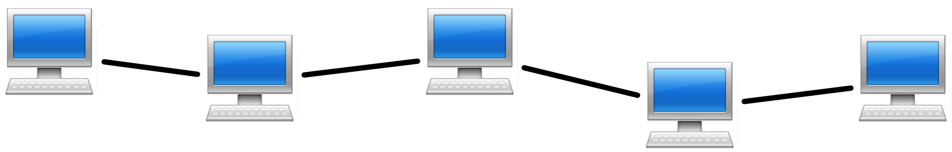
    - network = path 
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