Solutions to Assignment 4

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Claim 1. For each set A we have

$$m^*(A) = \inf\{\sum_{i\in\mathbb{N}} \mathrm{vol}(Q_i) : Q_1, Q_2, \dots \text{ are dyadic cubes with } A \subset \bigcup_{i\in\mathbb{N}} Q_i\}.$$

Proof. Let $A \subset \mathbb{R}^n$. Then for each $\varepsilon > 0$ there is an open set U with $A \subset U$ and $m^*(U) \leq m^*(A)+\varepsilon$. Now we know that each open set U can be written as an almost disjoint union of dyadic cubes $Q_1, Q_2, \dots$. That means $A \subset \bigcup_{i\in\mathbb{N}} Q_i$ and

$$\sum_{i\in\mathbb{N}} \mathrm{vol}(Q_i) = m^*(U) \leq m^*(A) + \varepsilon.$$

□

More direct proof. Consider the definition of the Lebesgue measure and use that for each $\varepsilon > 0$, each interval $I \subset \mathbb{R}^n$ can be covered by a finite collection of dyadic cubes $Q_1, \dots, Q_k$ with

$$\sum_{i=1}^{k} \mathrm{vol}(Q_k) \leq \mathrm{vol}(I) + \varepsilon.$$

□

Claim 2. For each cube Q there is a ball B with $Q \subset B$ and $m^*(B) \leq 4^d m^*(Q)$.

We don't prove this claim.

Now let $A \subset \mathbb{R}^n$. If $m^*(A) = 0$ then by Claim 1, for each $\varepsilon > 0$ there is a cover of A by cubes $Q_1, Q_2, \dots$ with

$$\sum_{i\in\mathbb{N}} m^*(Q_i) < \varepsilon.$$

By Claim 2 applied to each Q_i there is a cover of A by balls $B_1, B_2, \dots$ with

$$\sum_{i\in\mathbb{N}} m^*(B_i) < 4^d \varepsilon.$$

If A can be covered by balls $B_1, B_2, \ldots$ with

$$\sum_{i\in\mathbb{N}} m^*(B_i) < \varepsilon$$

then

$$m^*(A) \le \sum_{i\in\mathbb{N}} m^*(B_i) < \varepsilon.$$

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2.a

This follows from the fact that $\mathbb{Q}^n$ is countable and hence has Lebesgue measure zero, and from exercise 1.

Direct proof $\mathbb{Q}^n$ is countable, that is, we can enumerate it as $\mathbb{Q}^n = \{x^1, x^2, \ldots\}$. Let $\varepsilon > 0$. Then $(B(x^i, \varepsilon 2^{-i}))_{i\in\mathbb{N}}$ covers all of $\mathbb{Q}^n$ and

$$\sum_{i\in\mathbb{N}} m^*(B_i) = \varepsilon m^*(B(0,1)).$$

2.b

We know that for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ there is a $x^i \in \mathbb{Q}^n$ with $|x - x^i| < \varepsilon$. In particular, there is an i and a $y \in B(x^i, \varepsilon 2^{-i})$ with $|x - y| < \varepsilon$.

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Claim 3. For every $\varepsilon > 0$ there is a cube Q_ε with rational corner points and dyadic sidelengths and $\operatorname{diam}(Q_\varepsilon) \le 1$ and

$$m^*(Q_\varepsilon \setminus A) \le \varepsilon m^*(Q_\varepsilon).$$

Assuming this claim, we prove $m^*(\mathbb{R}^n \setminus B) = 0$. Let $\varepsilon > 0$ and Q_ε be the cube from Claim 3. Then there are rationals $q_1, q_2, \ldots$ such that

$$Q_0 := [0,1]^n = \bigcup_i (Q_\varepsilon + q_i)$$

where the union is almost disjoint. Thus

$$\begin{aligned} m^*(Q_0 \setminus B) &= \sum_i m^*((Q_\varepsilon + q_i) \setminus B) \\ &= \sum_i m^*(Q_\varepsilon \setminus (B - q_i)) \\ &= \sum_i m^*(Q_\varepsilon \setminus B)) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_i m^*(Q_\varepsilon \setminus A)) \\
&\le \varepsilon \sum_i m^*(Q_\varepsilon)) \\
&= \varepsilon m^*(Q_0).
\end{aligned}
$$

Letting $\varepsilon \to 0$ we obtain

$$m^*(Q_0 \setminus B) = 0.$$

Therefore

$$
\begin{aligned}
m^*(\mathbb{R}^n \setminus B) &= \sum_{q \in \mathbb{Z}^d} m^*((Q_0 + q) \setminus B) \\
&= \sum_{q \in \mathbb{Z}^d} m^*(Q_0 \setminus (B - q)) \\
&= \sum_{q \in \mathbb{Z}^d} m^*(Q_0 \setminus B) \\
&= 0
\end{aligned}
$$

Proof of Claim 3. Recall that it suffices to consider disjoint covers in the definition of the Lebesgue measure. So let $\varepsilon > 0$. There there are disjoint cubes $Q_1, Q_2, \ldots$ that cover A with

$$(1+\varepsilon) m^*(A) \ge \sum_i m^*(Q_i).$$

This is possible because we assumed $\varepsilon m^*(A) > 0$. Now there is at least one i with

$$(1+\varepsilon) m^*(Q_i \cap A) \ge m^*(Q_i)$$

because otherwise

$$\sum_i m^*(Q_i) > \sum_i (1+\varepsilon) m^*(A \cap Q_i) = (1+\varepsilon) m^*(A),$$

which is a contradiction to our assumption on the cover. That means

$$
\begin{aligned}
m^*(Q_i \setminus A) &= m^*(Q_i) - m^*(Q_i \cap A) \\
&\le \varepsilon m^*(Q_i \cap A) \\
&\le \varepsilon m^*(Q_i)
\end{aligned}
$$

Now inside of Q_i there is a cube Q with rational corners and dyadic dimensions and with

$$m^*(Q) \ge 3^{-d} m^*(Q_i).$$

Thus

$$
\begin{aligned}
m^*(Q \setminus A) &\le m^*(Q_i \setminus A) \\
&\le \varepsilon m^*(Q_i) \\
&\le 3^d \varepsilon m^*(Q).
\end{aligned}
$$

□

4

The function $f - g$ is continuous which means that for each $\varepsilon > 0$ the set $(f-g)^{-1}((\varepsilon, \infty)) = \{x : f(x) - g(x) > \varepsilon\}$ is open. Since $f = g$ almost everywhere, we have

$$m^*(\{x : f(x) - g(x) > \varepsilon\}) \le m^*(\{x : f(x) \ne g(x)\}) = 0,$$

and since the only open set with measure zero is the empty set, that means $\{x : f(x) - g(x) > \varepsilon\} = \emptyset$. Since it holds for all $\varepsilon > 0$ we get

$$\{x : f(x) - g(x) > 0\} = \emptyset.$$

Reversing the roles of f and g we also obtain

$$\{x : g(x) - f(x) > 0\} = \emptyset$$

so that we can conclude

$$\{x : g(x) \ne f(x)\} = \emptyset.$$

Alternative proof

Claim 4. Let $A \subset \mathbb{R}^n$ with $m^*(\mathbb{R}^n \setminus A) = 0$. Then A is dense in $\mathbb{R}^n$, that is for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ there is a $y \in A$ with $|x - y| < \varepsilon$.

Proof. Assume the contrary, i.e. there is an $x \in \mathbb{R}^n$ and an $\varepsilon > 0$ such that there is no $y \in A$ with $|x - y| < \varepsilon$. That means $B(x, \varepsilon) \subset \mathbb{R}^n \setminus A$ and thus

$$m^*(\mathbb{R}^n \setminus A) \ge m^*n(B(x, \varepsilon)) > 0,$$

a contradiction. □

Denote the set of points x with $f(x) = g(x)$ by A. Let $y \in \mathbb{R}^n$. Then by Claim 4 there is sequence $(x_i)_i$ with $x_i \in A$ and $x_i \to y$. Now by continuity

$$\begin{aligned} f(y) &= \lim_{i\to\infty} f(x_i) \\ &= \lim_{i\to\infty} g(x_i) \\ &= g(y). \end{aligned}$$

Thus $f = g$ on $\mathbb{R}^n$.

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5.a

Let $x \in \mathbb{R}^n$. Then

$$\lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} x/i = 0.$$

5.b

No. This follows from c.

Direct proof Assume that f is an almost uniform limit. Then there is a set A with $m^*(\mathbb{R}^n \setminus A) \leq 1$ and an N such that for all $x \in A$ and all $i \geq N$ we have

$$|f_i(x) - f(x)| \leq 1. \tag{1}$$

However A must contain a point x with $|x| \geq 6N$ so that

$$|f_N(x) - f_{2N}(x)| = |x|(\frac{1}{N} - \frac{1}{2N}) = \frac{|x|}{2N} \geq 3,$$

contradicting Equation (1).

5.c

No. Assume it converges to some function f in Lebesgue measure. Then there is a subsequence $(f_{i_k})_k$ that converges almost everywhere. By a) this means $f = 0$ almost everywhere. However for each ε we have

$$\begin{aligned} m^*(\{x : |f_{i_k}(x) - f(x)| > \varepsilon\}) &= m^*(\{x : |f_{i_k}(x)| > \varepsilon\}) \\ &= m^*((\frac{\varepsilon}{i_k}, \infty)) \\ &= \infty \end{aligned}$$

Alternative proof Let f be any function. We show that f_i cannot converge to f in measure. Now for each $x \geq 4i$ we have $f_i(x) - f_{2i}(x) = \frac{x}{2i} \geq 2$. Thus $|f_i(x) - f(x)| \geq 1$ or $|f_{2i}(x) - f(x)| \geq 1$. That means

$$[4i, \infty) \subset \{x : |f_i(x) - f(x)| \geq 1\} \cup \{x : |f_{2i}(x) - f(x)| \geq 1\}$$

and thus

$$m^*(\{x : |f_i(x) - f(x)| \geq 1\}) = \infty$$

or

$$m^*(\{x : |f_{2i}(x) - f(x)| \geq 1\}) = \infty$$

holds. This implies

$$\limsup_{i \to \infty} m^*(\{x : |f_i(x) - f(x)| \geq 1\}) = \infty,$$

making convergence to f in measure impossible.

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6.a

From the definition of the limit it follows that

$$\{x \in X : \lim_{i \to \infty} f_i(x) \neq f(x)\} = \{x \in X : \exists k \in \mathbb{N}\ \forall j\ \exists i \geq j |f_i(x) - f(x)| > \frac{1}{k}\}$$

$$
\begin{aligned}
&= \bigcup_{k\in\mathbb{N}} \{x \in X : \forall j \ \exists i \geq j |f_i(x) - f(x)| > \frac{1}{k}\} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \{x \in X : \exists i \geq j |f_i(x) - f(x)| > \frac{1}{k}\} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \underbrace{\{x \in X : \sup_{i\geq j} |f_i(x) - f(x)| > \frac{1}{k}\}}_{=:A_{>\frac{1}{k}}} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} A_{>\frac{1}{k}} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} A_{\geq\frac{1}{k}} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \{x \in X : \sup_{i\geq j} |f_i(x) - f(x)| \geq \frac{1}{k}\}
\end{aligned}
$$

For the last step, $A_{>\frac{1}{k}} \subset A_{\geq\frac{1}{k}}$ is clear. The other direction follows from $A_{\geq\frac{1}{k}} \subset A_{>\frac{1}{k+1}}$.

6.b

Assume that for almost every $x \in X$ we have $f_i(x) \to f(x)$. Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ with $\frac{1}{n} \leq \varepsilon$. Then

$$
\begin{aligned}
m^*(\bigcap_{j\in\mathbb{N}} A_{\geq\varepsilon}) &\leq m^*(\bigcap_{j\in\mathbb{N}} A_{\geq\frac{1}{n}}) \\
&\leq m^*(\bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} A_{\frac{1}{k}})
\end{aligned}
$$

and by a) we have

$$
= 0
$$