Solutions to Assignment 5

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1

If f is measurable then by definition in particular for every rational q the set $\{x \in X : f(x) > q\}$ is measurable.

For the reverse direction, note that for every $r \in \mathbb{R}$ we have

$$\{x \in X : f(x) > r\} = \bigcup_{x \in \mathbb{Q},\ q \geq r} \{x \in X : f(x) > q\}.$$

Now if each set in the union on the right hand side is measurable, then so is the left hand side.

2

If f is measurable, then so is

$$f_i = f 1_{-i < f < i} f + i 1_{f \geq i} - i 1_{f \leq -i}.$$

For the reverse direction, let $r \in \mathbb{R}$ and take $i > |r|$. Then

$$\{x \in X : f(x) > r\} = \{x \in X : f_i(x) > r\}$$

which is measurable by assumption. Alternatively it also follows from the fact that f is the pointwise limit of the sequence $(f_i)_i$.

3

It suffices to show that $\{A \subset \mathbb{R}^n : f(A) \text{ is Borel}\}$ is

1. a σ-algebra
2. that contains all open sets,

because that means in particular all Borel sets satisfy that $f(A)$ is Borel.

1. In order to show that the set it is a σ-algebra, let $f(A)$ be Borel. Then $f(A^\complement) = f(A)^\complement$ is Borel. And if $f(A_1), f(A_2), \ldots$ are Borel, then $f(A_1 \cup A_2 \cup \ldots) = f(A_1) \cup f(A_2) \cup \ldots$ is Borel.

2. The set contains all open sets, since for any open A we have

$$f(A) = (f^{-1})^{-1}(A)$$

which is open, since f^{-1} is continuous, and the preimage of an open set under continuous functions is open.

4

Since the pointwise limit of measurable functions is measurable, and zero measurable sets are measurable, a pointwise a.e. limit of continuous functions is measurable.

For the other direction, let f be measurable. Then

$$f = f^+ - f_-$$

where f^+ and f^- are measurable and nonnegative. So if we approximate f^+ and f^- by continuous function, then their sum will tend to f almost everywhere. That means it suffices to consider nonnegative functions f.

Furthermore it is true for a.e. x that

$$f(x) = \sum_{z \in \mathbb{Z}^n} 1_{(0,1)^n+z}(x) f(x).$$

Now if we approximate each $f 1_{(0,1)^n+z}$ by continuous functions that are supported on $(0,1)^n + z$, their sum will tend to f a.e.. Hence after translation, it suffices to consider f supported on $(0,1)^n$.

Now we write

$$f = f 1_{f=\infty} + f 1_{f<\infty}.$$

Lemma 1. Let $A \subset (0,1)^n$ be measurable. Then for every $\varepsilon > 0$ there is a compact set K_ε and an open set $U_\varepsilon \subset (0,1)^d$ with $K_\varepsilon \subset A \subset U_\varepsilon$ and there is a continuous function g_ε that is 0 outside of U_ε and 1 on K_ε.

Proof. We already know this from the lecture. □

Applying Lemma 1 to $A = \{x : f(x) = \infty\}$, we obtain a sequnce of functions $(k g_{2^{-k}})_k$ that are supported in $(0,1)^n$ and converge to $f 1_{f=\infty}$ a.e. since $\sum_{k \in \mathbb{N}} 2^{-k} < \infty$.

Now we only need to approximate $f 1_{f<\infty}$ by functions supported in $(0,1)^n$, which means it suffices to consider the case that f is finite everrywhere.

Claim 2. There is a sequence of functions $(f_k)_k$ that converges to f in measure.

A sequence of functions that converges in measure has a subsequence that converges almost everywhere, hence it remains to prove Claim 2.

Proof of Claim 2. Let $\varepsilon > 0$. Then there is an N with $m^*(\{x : f(x) > N\}) < \varepsilon$ and a simple function $\sum_i a_i 1_{A_i}$ that differs from $f 1_{f \leq N}$ by at most ε. We apply Lemma 1 to each A_i with $\varepsilon 2^{-i}$ and obtain a continuous function

$$f_\varepsilon := \sum_i a_i g_{i,\varepsilon}$$

that is equal to $f1_{f\leq N}$ everywhere except on a set of measure at most ε and is supported on $(0,1)^n$. That means

$$m^*(\{x : |f_\varepsilon(x) - f(x)| > \varepsilon\}) \leq m^*(\{x : f(x) > N\}) + m^*(\{x : f_\varepsilon(x) \neq \sum_i a_i 1_{A_i} 1_{f\leq N}\}) \leq 2\varepsilon.$$

□

5

5.a

Let A be the set of points where f_i converges to f. Then for each i the function $f_i 1_A$ is measurable, and the functions $f_i 1_A$ converge to $f 1_A$ everywhere pointwise. Hence $f 1_A$ is measurable. Since $\mu(X \setminus A) = 0$, also the function $f = f1_A + f1_{X\setminus A}$ is measurable.

5.b

Let f^1, f^2 be two function such that for $j = 1, 2$ we have almost everywhere pointwise $f_i \to f^j$. Let A^j be the set where on which f_i converges to f pointwise. Then $f^1 = f^2$ on $A^1 \cap A^2$, and

$$\mu(X \setminus (A^1 \cap A^2)) \leq \mu(X \setminus A^1) + \mu(X \setminus A^2) = 0.$$

Hence $f^1 = f^2$ almost everywhere.

5.c

Let

$$A := \{x \in X : f_i(x) \not\to f(x)\} \cup \{x \in X : g(x) \neq f(x)\} \cup \bigcup_i \{x \in X : f_i(x) \neq g_i(x)\}.$$

Then A has measure 0 and $g_i = f_i$ and $g = f$ and $f_i \to f$ on $X \setminus A$. Hence $g_i \to g$ on $X \setminus A$.

6

Let $\varepsilon > 0$. Then

$$\begin{aligned}\{x \in X : |[f_i(x) + g_i(x)] - [f(x) + g(x)]| > \varepsilon\} \subset \{x \in X : |f_i(x) - f(x)| > \frac{\varepsilon}{2}\} \\ \cup \{x \in X : |g_i(x) - g(x)| > \frac{\varepsilon}{2}\}\end{aligned}$$

which implies

$$\begin{aligned}\limsup_{i\to\infty} \mu(\{|(f_i + g_i) - (f + g)| > \varepsilon\}) &\leq \limsup_{i\to\infty} \mu(\{|f_i - f| > \frac{\varepsilon}{2}\}) \\ &\quad + \limsup_{i\to\infty} \mu(\{|g_i - g| > \frac{\varepsilon}{2}\}) \\ &= 0\end{aligned}$$