1. **Gaussin ja Stokesin lauseet**

$$\oint_{\partial V} d\vec{S} * \vec{A} = \int_V \nabla * \vec{A}\, dV, \quad * = \cdot,\ \times \text{ tai ” ”} \tag{1}$$

$$\oint_{\partial S} d\vec{l} * \vec{A} = \int_S (d\vec{S} \times \nabla) * \vec{A} \tag{2}$$

2. **Greenin kaavat**

$$\text{I}: \quad \int_V \nabla\psi \cdot \nabla\xi \; dV + \int_V \psi\nabla^2\xi \; dV = \oint_{\partial V} \psi\nabla\xi \cdot d\vec{S} \tag{3}$$

$$\text{II}: \quad \int_V \left(\psi\nabla^2\xi - \xi\nabla^2\psi\right) \; dV = \oint_{\partial V} (\psi\nabla\xi - \xi\nabla\psi) \cdot d\vec{S} \tag{4}$$

3. **Diracin deltafunktio**

$$\int_{V'} f(\vec{r}\,')\delta(\vec{r} - \vec{r}\,')dV\,' = f(\vec{r}) \tag{5}$$

4. **Vektorilaskentaa**

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \tag{6}$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A} \tag{7}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \tag{8}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{9}$$

$$\vec{A} \cdot \overline{\overline{I}} = \vec{A} \tag{10}$$

$$\nabla(\psi\xi) = \xi\nabla\psi + \psi\nabla\xi \tag{11}$$

$$\nabla \times (\psi\vec{A}) = \nabla\psi \times \vec{A} + \psi\nabla \times \vec{A} \tag{12}$$

$$\nabla \cdot (\psi\vec{A}) = \nabla\psi \cdot \vec{A} + \psi\nabla \cdot \vec{A} \tag{13}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \tag{14}$$

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \tag{15}$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \tag{16}$$

$$\vec{A} \times (\nabla \times \vec{A}) = \frac{1}{2}\nabla(A^2) - (\vec{A} \cdot \nabla)\vec{A} \tag{17}$$

$$\nabla \cdot (\vec{A}\vec{B}) = (\nabla \cdot \vec{A})\vec{B} + \vec{A} \cdot \nabla\vec{B} \tag{18}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \tag{19}$$

$$\nabla \times \nabla\psi = 0 \tag{20}$$

$$\nabla \cdot \nabla \times \vec{A} = 0 \tag{21}$$

$$\nabla r = \frac{\vec{r}}{r} \quad (22)$$

$$\nabla \vec{r} = \overline{\overline{I}} \quad (23)$$

$$\nabla \cdot \vec{r} = \dim \vec{r} \quad (24)$$

$$\nabla \times \vec{r} = \vec{0},\ \vec{r} \in \Re^3 \quad (25)$$

$$\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = -4\pi\delta(\vec{r} - \vec{r}\,') \quad (26)$$

5. **Trigonometriaa**

$$\sin^2 x + \cos^2 x = 1 \quad (27)$$

$$\sin(2x) = 2\sin x \cos x \quad (28)$$

$$\cos(2x) = \cos^2 x - \sin^2 x \quad (29)$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \quad (30)$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \quad (31)$$

$$e^{\pm ix} = \cos x \pm i \sin x \quad (32)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad (33)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad (34)$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \quad (35)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \quad (36)$$

6. **Integraaleja**

$$\int_a^b A dB = AB|_a^b - \int_a^b B dA \quad (37)$$

$$\int_V (\nabla\psi \cdot \vec{A}) dV = \oint_{\partial V} (\psi \vec{A}) \cdot d\vec{S} - \int_V \psi \nabla \cdot \vec{A} dV \quad (38)$$

$$\int_S \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \cdot d\vec{S} = \Omega_S(\vec{r}\,') \quad \text{(avaruuskulma)} \quad (39)$$

$$\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 + x^2}} \quad (40)$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right) = \operatorname{arsin}\left(\frac{x}{a}\right) \quad (41)$$

$$\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1} \quad (42)$$

$$\int \sin^2(ax) dx = \frac{1}{2}x - \frac{1}{4a}\sin(2ax) \quad (43)$$

$$\int \cos^2(ax) dx = \frac{1}{2}x + \frac{1}{4a}\sin(2ax) \quad (44)$$

7. **Taylorin sarja**

$$f(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left((\vec{r}-\vec{a})\cdot\nabla\right)^n f(\vec{r})\Big|_{\vec{r}=\vec{a}} \tag{45}$$

$$f(\vec{r}+\vec{h}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\vec{h}\cdot\nabla\right)^n f(\vec{r}) \tag{46}$$

$$\tag{47}$$

$$(1+\epsilon)^n = 1 + n\epsilon + \frac{n(n-1)\epsilon^2}{2!} + \frac{n(n-1)(n-2)\epsilon^3}{3!} + \ldots, \quad n \in \mathbb{Z}_+ \text{ tai } \epsilon < 1 \tag{48}$$

$$\begin{aligned} \frac{1}{|\vec{r}-\vec{r}\,'|} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\vec{r}\,'\cdot\nabla)^n \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\Big|_{\vec{r}\,'=0} \\ &= \frac{1}{|\vec{r}-\vec{r}\,'|}\Big|_{\vec{r}\,'=0} + \sum_{i=1}^{3} x_i' \frac{\partial}{\partial x_i} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\Big|_{\vec{r}\,'=0} \\ &\quad -\frac{1}{2} \sum_{i=1}^{3}\sum_{j=1}^{3} x_i' x_j' \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\Big|_{\vec{r}\,'=0} + \ldots, \quad r > r\,' \end{aligned} \tag{49}$$

8. **Konformisia kuvauksia**

Schwarzin transformaatio:

$$\frac{dz}{df} = \frac{C}{(f-a_1)^{k_1}\ldots(f-a_n)^{k_n}} \tag{50}$$

Möbiuksen kuvaus:

$$w = \frac{az+b}{cz+d}, \quad (ad-bc \neq 0) \tag{51}$$

9. **Käyräviivaiset koordinaatistot**

Sylinterikoordinaatisto (ρ, φ, z):

$$\vec{r} = \rho\vec{e}_\rho + \varphi\vec{e}_\varphi + z\vec{e}_z \Rightarrow d\vec{r} = \vec{e}_\rho\, d\rho + \vec{e}_\varphi \rho\, d\varphi + \vec{e}_z\, dz \Rightarrow \quad dV = \rho\, d\rho\, d\varphi\, dz \tag{52}$$

Pallokoordinaatisto (r, θ, φ):

$$\vec{r} = r\vec{e}_r + \theta\vec{e}_\theta + \varphi\vec{e}_\varphi \Rightarrow d\vec{r} = \vec{e}_r\, dr + \vec{e}_\theta r\, d\theta + \vec{e}_\varphi r\sin\theta\, d\varphi \Rightarrow \quad dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi \tag{53}$$

Paikkakoordinaattien muunnokset:

$$\begin{aligned} x &= \rho\cos\varphi = r\sin\theta\cos\varphi \\ y &= \rho\sin\varphi = r\sin\theta\sin\varphi \\ z &= z = r\cos\theta \end{aligned} \tag{54}$$

$$\begin{aligned} \sqrt{x^2+y^2} &= \rho = r\sin\theta \\ \arctan(y/x) &= \varphi = \varphi \\ z &= z = r\cos\theta \end{aligned} \tag{55}$$

$$
\begin{array}{ccccc}
\sqrt{x^2+y^2+z^2} & = & \sqrt{\rho^2+z^2} & = & r \\
\arctan(\sqrt{x^2+y^2}/z) & = & \arctan(\rho/z) & = & \theta \\
\arctan(y/x) & = & \varphi & = & \varphi
\end{array} \tag{56}
$$

Yksikkövektorien muunnokset:

$$
\begin{array}{ccccc}
\vec{e}_x & = & \cos\varphi\vec{e}_\rho - \sin\varphi\vec{e}_\varphi & = & \sin\theta\cos\varphi\vec{e}_r + \cos\theta\cos\varphi\vec{e}_\theta - \sin\varphi\vec{e}_\varphi \\
\vec{e}_y & = & \sin\varphi\vec{e}_\rho + \cos\varphi\vec{e}_\varphi & = & \sin\theta\sin\varphi\vec{e}_r + \cos\theta\sin\varphi\vec{e}_\theta - \cos\varphi\vec{e}_\varphi \\
\vec{e}_z & = & \vec{e}_z & = & \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta
\end{array} \tag{57}
$$

$$
\begin{array}{ccccc}
\cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y & = & \vec{e}_\rho & = & \sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta \\
-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y & = & \vec{e}_\varphi & = & \vec{e}_\varphi \\
\vec{e}_z & = & \vec{e}_z & = & \cos\theta\vec{e}_r - \sin\theta\vec{e}_\theta
\end{array} \tag{58}
$$

$$
\begin{array}{ccccc}
\sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z & = & \sin\theta\vec{e}_\rho + \cos\theta\vec{e}_z & = & \vec{e}_r \\
\cos\theta\cos\varphi\vec{e}_x + \cos\theta\sin\varphi\vec{e}_y - \sin\theta\vec{e}_z & = & \cos\theta\vec{e}_\rho - \sin\theta\vec{e}_z & = & \vec{e}_\theta \\
-\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y & = & \vec{e}_\varphi & = & \vec{e}_\varphi
\end{array} \tag{59}
$$

10. **Gradientti eri koordinaatistoissa.**

Karteesinen:

$$\nabla\psi(x,y,z) = \vec{e}_x\frac{\partial\psi}{\partial x} + \vec{e}_y\frac{\partial\psi}{\partial y} + \vec{e}_z\frac{\partial\psi}{\partial z} \tag{60}$$

Sylinteri:

$$\nabla\psi(\rho,\varphi,z) = \vec{e}_\rho\frac{\partial\psi}{\partial\rho} + \vec{e}_\varphi\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \vec{e}_z\frac{\partial\psi}{\partial z} \tag{61}$$

Pallo:

$$\nabla\psi(r,\theta,\varphi) = \vec{e}_r\frac{\partial\psi}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \vec{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi} \tag{62}$$

11. **Divergenssi eri koordinaatistoissa.**

Karteesinen:

$$\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{63}$$

Sylinteri:

$$\nabla\cdot\vec{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z} \tag{64}$$

Pallo:

$$\nabla\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\varphi}{\partial\varphi}\right] \tag{65}$$

12. **Roottori eri koordinaatistoissa.**

Karteesinen:

$$\nabla\times\vec{V} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \tag{66}$$

Sylinteri:

$$\nabla \times \vec{V} = \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix} \tag{67}$$

Pallo:

$$\nabla \times \vec{V} = \frac{1}{r^2\sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ V_r & rV_\theta & r\sin\theta V_\varphi \end{vmatrix} \tag{68}$$

13. **Laplace eri koordinaatistoissa.**

Karteesinen:

$$\nabla^2\psi(x,y,z) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{69}$$

Sylinteri:

$$\nabla^2\psi(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{70}$$

Pallo:

$$\nabla^2\psi(r,\theta,\varphi) = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] \tag{71}$$

14. **Laplacen yhtälön yleisiä ratkaisuja**

Karteesinen:

$$V(x,y,z) = (A\sin lx + B\cos lx)(C\sin my + D\cos my)(E\sinh nz + F\cosh nz) \tag{72}$$

$$V(x,y,z) = (A\sin lx + B\cos lx)(C\sin my + D\cos my)(Ee^{nz} + Fe^{-nz}), \tag{73}$$

missä $n^2 = l^2 + m^2$.

Sylinteri:

$$\begin{aligned} V(\rho,\varphi,z) = {} & [AJ_m(\lambda\rho) + BN_m(\lambda\rho)][C\sin m\varphi + D\cos m\varphi][E\sinh\lambda z + F\cosh\lambda z] \\ & +[GI_m(\lambda\rho) + HK_m(\lambda\rho)][R\sin m\varphi + S\cos m\varphi][T\sin\lambda z + U\cos\lambda z] \end{aligned} \tag{74}$$

z-riippumaton tapaus (napakoordinaatit)

$$V(\rho,\varphi) = E + F\ln\rho + (A\rho^n + B\rho^{-n})(C\cos n\varphi + D\sin n\varphi) \tag{75}$$

Pallo:

$$V(r,\theta,\varphi) = [Ar^l + Br^{-(l+1)}][C\sin m\varphi + D\cos m\varphi]P_l^m(\cos\theta) \tag{76}$$

$$V(r,\theta,\varphi) = [Ar^l + Br^{-(l+1)}][Ce^{im\varphi} + De^{-im\varphi}]P_l^m(\cos\theta) \tag{77}$$

$$V(r,\theta,\varphi) = [A_{l,m}r^l + B_{l,m}r^{-(l+1)}]Y_l^m(\theta,\varphi) \tag{78}$$

φ-riippumaton tapaus:

$$V(r,\theta) = [A_lr^l + B_lr^{-(l+1)}]P_l(\cos\theta) \tag{79}$$

15. **Sini- ja kosinifunktioiden ortogonaalisuus**

Seuraavissa kaavoissa n, m ja k ovat kokonaislukuja väliltä $[1, \infty]$.

$$\int_0^{\frac{k\pi}{2}} \sin nx \sin mx \; dx = \begin{cases} \frac{k\pi}{4} & , n = m \\ 0 & , n \neq m \end{cases} \tag{80}$$

$$\int_0^{\frac{k\pi}{2}} \cos nx \cos mx \; dx = \begin{cases} \frac{k\pi}{4} & , n = m \\ 0 & , n \neq m \end{cases} \tag{81}$$

$$\int_0^{k\pi} \sin nx \cos mx \; dx = \begin{cases} \frac{2n}{n^2 - m^2} & , k \text{ pariton ja } (n+m) \text{ pariton} \\ 0 & , k \text{ pariton ja } (n+m) \text{ parillinen} \\ 0 & , k \text{ parillinen} \end{cases} \tag{82}$$

16. **Fourier-sarjat**

Fourier-sarja $2L$-jaksollisille funktioille:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{83}$$

missä

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \; dx, \qquad n = 0, 1, 2, 3, \ldots \tag{84}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \; dx, \qquad n = 1, 2, 3, \ldots \tag{85}$$

Fourier-kosinisarja $2L$-jaksollisille parillisille funktioille:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \tag{86}$$

missä

$$a_n = \frac{2}{L} \int_0^{L} f(x) \cos \frac{n\pi x}{L} \; dx, \qquad n = 0, 1, 2, 3, \ldots \tag{87}$$

Fourier-sinisarja $2L$-jaksollisille parittomille funktioille:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \tag{88}$$

missä

$$b_n = \frac{2}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{L} \; dx, \qquad n = 1, 2, 3, \ldots \tag{89}$$

2–ulotteinen Fourier–sarja

$$f(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right), \tag{90}$$

$$A_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \, dy \, dx \tag{91}$$

17. **Legendren polynomit $P_n(x)$**

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2-1)^n \tag{92}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos\gamma), \quad \gamma = \sphericalangle(\vec{r}_<, \vec{r}_>) \tag{93}$$

$$(1 \mp 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} (\pm 1)^n P_n(x) t^n \tag{94}$$

Legendren differentiaaliyhtälöt:

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP_n(\cos\theta)}{d\theta}\right) + n(n+1) P_n(\cos\theta) = 0 \tag{95}$$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \tag{96}$$

Rekursiokaavoja:

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x), \quad n = 1,2,3,\ldots \tag{97}$$

$$P_{n+1}'(x) + P_{n-1}'(x) = 2x P_n'(x) + P_n(x), \quad n = 1,2,3,\ldots \tag{98}$$

Ortogonaalisuus:

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{2\delta_{mn}}{2n+1} \tag{99}$$

Pariteetti:

$$P_n(-x) = (-1)^n P_n(x) \tag{100}$$

Arvoja:

$$|P_n(\cos\theta)| \leq 1 \tag{101}$$

$$P_n(1) = 1 \tag{102}$$

$$P_0(x) = 1 \tag{103}$$

$$P_1(x) = x \tag{104}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{105}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{106}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{107}$$

18. **Legendren liittopolynomit $P_n^m(x)$**

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \quad m \le n \tag{108}$$

Legendren assosioidut differentiaaliyhtälöt:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_n^m(\cos\theta)}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] P_n^m(\cos\theta) = 0 \tag{109}$$

$$(1-x^2)\frac{d^2}{dx^2}P_n^m(x) - 2x\frac{d}{dx}P_n^m(x) + \left[n(n+1) - \frac{m^2}{1-x^2}\right] P_n^m(x) = 0 \tag{110}$$

Rekursiokaavoja:

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x) \tag{111}$$

$$(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m \tag{112}$$

$$P_n^{m+1} = \frac{2mx}{(1-x^2)^{\frac{1}{2}}} P_n^m - [n(n+1) - m(m-1)]P_n^{m-1} \tag{113}$$

Ortogonaalisuus:

$$\int_{-1}^{1} P_p^m(x)P_q^m(x)\ dx = \frac{2}{2q+1}\frac{(q+m)!}{(q-m)!}\delta_{pq} \tag{114}$$

$$\int_0^{\pi} P_p^m(\cos\theta)P_q^m(\cos\theta)\sin\theta\ d\theta = \frac{2}{2q+1}\frac{(q+m)!}{(q-m)!}\delta_{pq} \tag{115}$$

Pariteetti:

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x) \tag{116}$$

Arvoja:

$$P_n^m(\pm 1) = 0, \quad m \ne 0 \tag{117}$$

$$P_1^1(x) = (1-x^2)^{\frac{1}{2}} = \sin\theta \tag{118}$$

$$P_2^1(x) = 3x(1-x^2)^{\frac{1}{2}} = 3\cos\theta\sin\theta \tag{119}$$

$$P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}} = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta \tag{120}$$

$$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta \tag{121}$$

Legendren funktioiden kuvaajia on esitetty kokoelman lopussa kuvassa 1.

19. **Palloharmoniset funktiot $Y_n^m(\theta,\varphi)$**

$$Y_n^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta)e^{im\varphi}, \quad -n \le m \le n \tag{122}$$

Differentiaaliyhtälö:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n^m(\theta,\varphi)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n^m(\theta,\varphi)}{\partial\varphi^2}+n(n+1)Y_n^m(\theta,\varphi)=0 \tag{123}$$

Ortogonaalisuus:

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{n_1}^{m_1*}(\theta,\varphi)Y_{n_2}^{m_2}(\theta,\varphi)\sin\theta\ d\theta d\varphi=\delta_{n_1n_2}\delta_{m_1m_2} \tag{124}$$

Yhteenlaskuteoreema:

$$P_n(\cos\gamma)=\frac{4\pi}{2n+1}\sum_{m=-n}^{n}Y_n^m(\theta_1,\varphi_1)Y_n^{m*}(\theta_2,\varphi_2), \tag{125}$$

missä γ on suuntien (θ_1,φ_1) ja (θ_2,φ_2) välinen kulma.

Kompleksikonjugaatti:

$$Y_n^{m*}(\theta,\varphi)=(-1)^mY_n^{-m}(\theta,\varphi) \tag{126}$$

Arvoja:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \tag{127}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{128}$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ e^{\pm i\varphi} \tag{129}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right) \tag{130}$$

$$Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{5}{24\pi}}\,3\sin\theta\cos\theta\ e^{\pm i\varphi} \tag{131}$$

$$Y_2^{\pm2}(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\theta\ e^{\pm 2i\varphi} \tag{132}$$

20. Besselin funktiot $J_n(x)$ ja Neumannin funktiot $N_n(x)$

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} \tag{133}$$

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \tag{134}$$

Besselin differentiaaliyhtälö:

$$x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)J_n(x)=0 \tag{135}$$

Differentiaaliyhtälön ratkaisuja ovat myös Neumannin funktiot $N_n(x)$:

$$N_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \tag{136}$$

Rekursiokaavoja:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{137}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \tag{138}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \tag{139}$$

Myös Neumannin funktiot toteuttavat yo. rekursiokaavat.

Ortogonaalisuus:

$$\int_0^a J_n\left(\rho_{ni}\frac{r}{a}\right) J_n\left(\rho_{nj}\frac{r}{a}\right) r\ dr = \frac{a^2}{2}\left[J_{n+1}(\rho_{ni})\right]^2 \delta_{ij}, \tag{140}$$

missä ρ_{ni} on funktion J_n i:s nollakohta.

Sarjaesitykset:

$$\cos x = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x) \tag{141}$$

$$\sin x = 2\sum_{k=1}^{\infty}(-1)^{k+1} J_{2k-1}(x) \tag{142}$$

Yhteenlaskukaava:

$$J_n(x_1 + x_2) = \sum_{m=-\infty}^{\infty} J_m(x_1) J_{n-m}(x_2) \tag{143}$$

Muita kaavoja:

$$|J_n(x)| \leq 1 \tag{144}$$

$$J_{-n}(x) = (-1)^n J_n(x) \tag{145}$$

Besselin funktioiden kuvaajia on esitetty kokoelman lopussa kuvassa 2.

21. Muita erikoisfunktioita

Modifioidut Besselin funktiot $I_n(x)$ ja $K_n(x)$:

$$I_n(x) = -i^n J_n(ix) \tag{146}$$

$$K_n(x) = \frac{\pi i^{n+1}}{2} H_n^{(1)}(ix) \tag{147}$$

I_n ja K_n ratkaisevat differentiaaliyhtälön:

$$x^2 f'' + x f' - (x^2 + n^2) f = 0 \tag{148}$$

Funktiot K_n divergoivat pisteessä $x = 0$, ja funktiot I_n divergoivat, kun $x \to \infty$.

Hankelin Funktiot $\boldsymbol{H_n(x)}$:

$$H_n^{(1)}(x) = J_n(x) + i N_n(x) \tag{149}$$

$$H_n^{(2)}(x) = J_n(x) - i N_n(x) \tag{150}$$

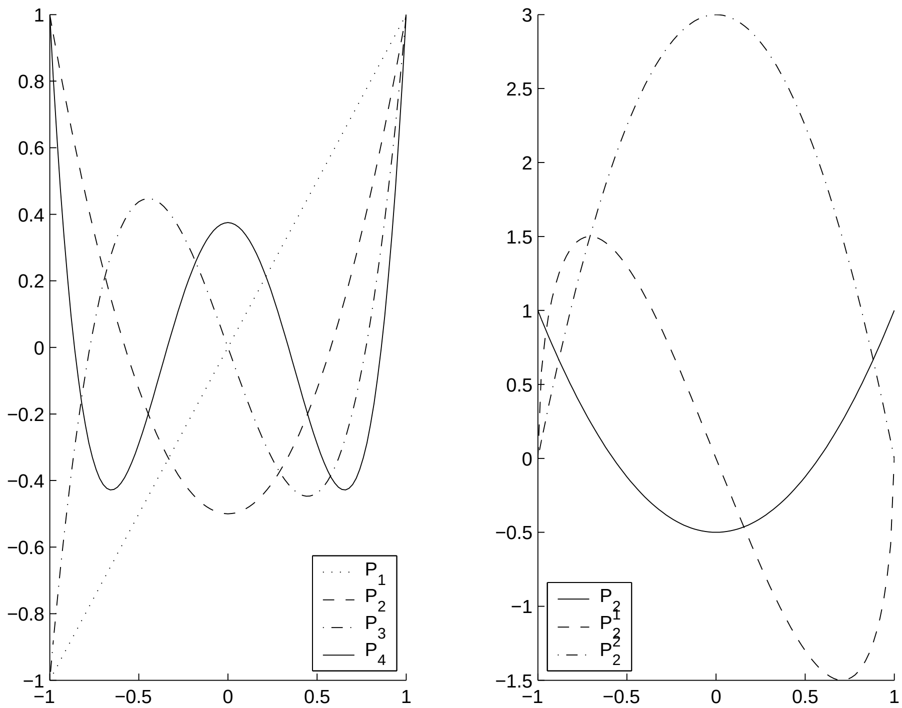
Besselin pallofunktiot $\boldsymbol{j_l(x)}$ ja $\boldsymbol{n_l(x)}$:

$$j_l(x) = \sqrt{\frac{\pi}{2x}}\, J_{l+\frac{1}{2}}(x) \tag{151}$$

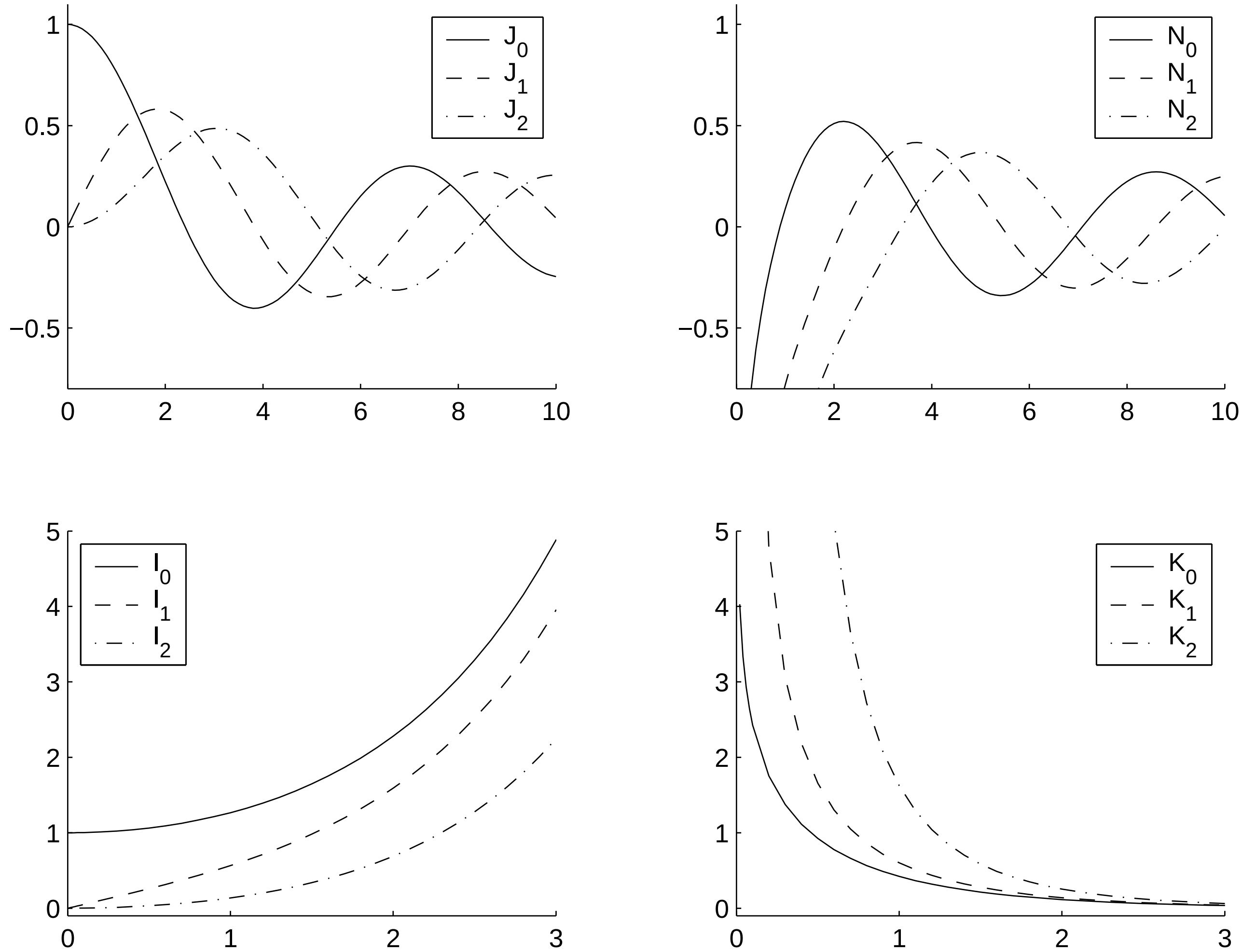
$$n_l(x) = \sqrt{\frac{\pi}{2x}}\, N_{l+\frac{1}{2}}(x) \tag{152}$$

j_l ja n_l ratkaisevat differentiaaliyhtälön:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) f + \left[1 - \frac{l(l-1)}{r^2}\right] f = 0 \tag{153}$$



Kuva 1: Legendren polynomien kuvaajia.



Kuva 2: Besselin funktioiden kuvaajia.