ELEC-C8201 Control and Automation

Lecture 3: Poles and zeros, system speed, stability and oscillations

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System behavior

- The most important feature of the system is stability. Other features include:
  - Speed
  - Oscillations and their nature, resonance
  - Minimum phase/non minimum phase, start transients
  - Sensitivity and robustness, Disturbance rejection
  - Possible integral or derivative behavior
  - Degree of TF (strictly proper, proper, not proper)

- In previous lectures, the behavior of the system was determined accurately from the responses – which required the inverse Laplace transform. This lecture focuses on behavioral analysis based on model properties without the analytical calculation of the response function.

System Poles and Zeros

- The transfer function consists of the numerator and denominator polynomial
- $$G(s) = \frac{b_1 s^{n_z-1} + b_2 s^{n_z-2} + \cdots + b_{n_z-2} s^2 + b_{n_z-1} s + b_{n_z}}{s^{n_p} + a_1 s^{n_p-1} + a_2 s^{n_p-2} + \cdots + a_{n_p-2} s^2 + a_{n_p-1} s + a_{n_p}} = \frac{P(s)}{Q(s)}$$
- The denominator polynomial is the characteristic polynomial of the system

  Characteristic polynomial: Q(s)

  Characteristic equation: Q(s)=0
- Polynomial can be written as a product of its roots:

$$G(s) = \frac{b_1 (s - s_{z1})(s - s_{z2}) \cdots (s - s_{z(n_z-1)})(s - s_{zn_z})}{(s - s_{p1})(s - s_{p2}) \cdots (s - s_{p(n_p-1)})(s - s_{pn_p})}$$

- The values for which numerator is zeros(roots) are the zeros of the system and those for which denominator is zero (roots) are the poles of the system.

System Poles and Zeros

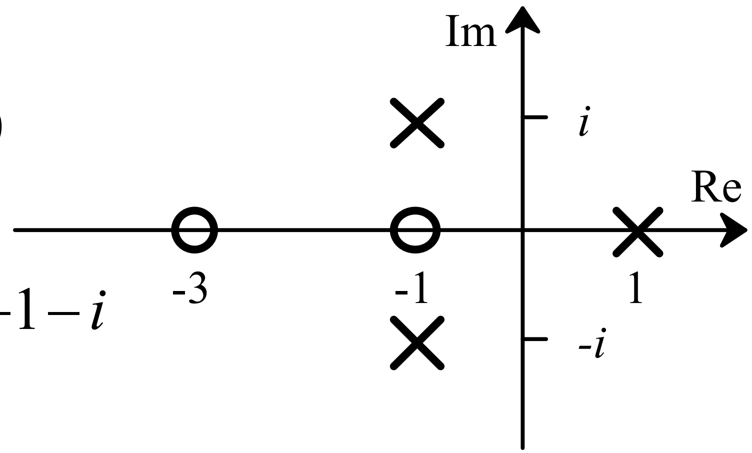
- In the polar representation, the poles and zeros of the system are plotted graphically in a complex plane:

$$G(s) = \frac{(s+1)(s+3)}{(s-1)(s^2+2s+2)} = \frac{(s+1)(s+3)}{(s-1)((s+1)^2+1)} = \frac{(s+1)(s+3)}{(s-1)(s+1+i)(s+1-i)}$$

Characteristic equation: $(s-1)(s^2+2s+2) = 0$

System zeros: $s_{z1} = -1, s_{z2} = -3$

System poles: $s_{p1} = 1, s_{p2} = -1+i, s_{p3} = -1-i$

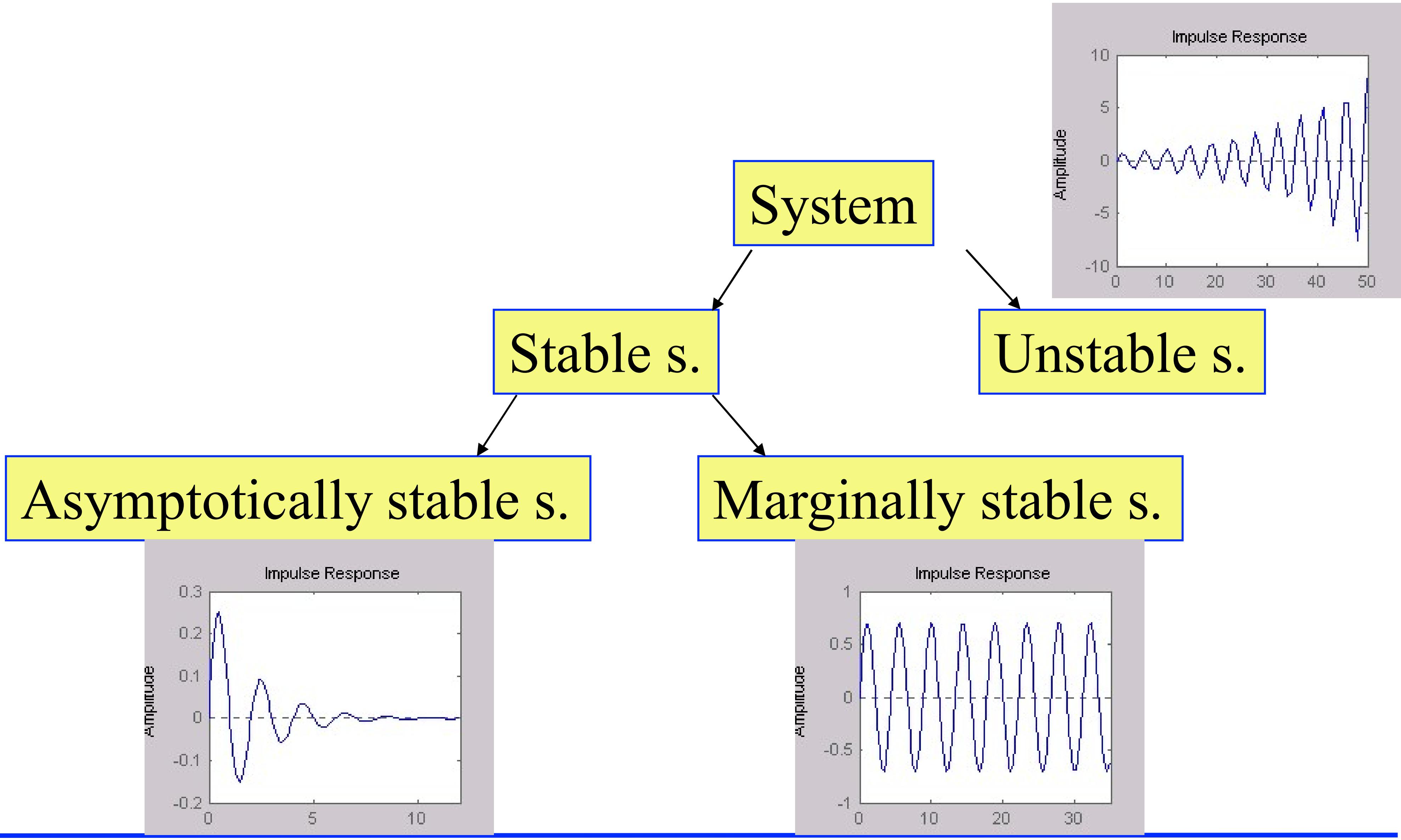


- Complex roots always occur in complex conjugate pairs: $s_i = a \pm bi$ => The complex poles are always symmetric wrt the real axis.

Stability

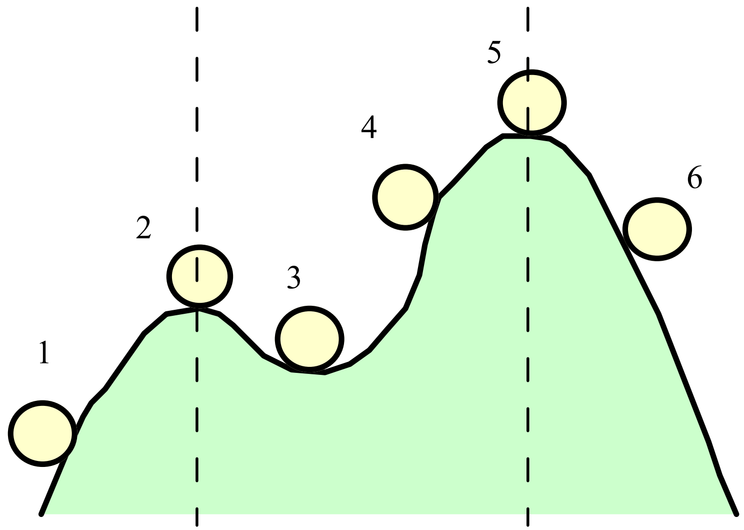
- Several different definitions have been developed for stability. E.g.
  - Stability of single solution (nonlinear and/or time varying systems)
  - System stability (global feature of linear systems)
  - Global stability vs. local stability (non-linear systems)
  - Lyapunov stability
  - Asymptotic stability
  - BIBO-stability (Bounded Input - Bounded Output)
  - Marginal stability
- This course covers only the stability of linear systems, in which case stability is a global feature of the system – Stability does not depend on the input quantities or the operational region.

System stability



Stability

- The non-linear system stability may depend on the operating range and the input quantities. Consider, for example, rocks on a mountainside.
  - The microdermabrasion has one stable equilibrium point (Valley, 3) and two unstable (peaks, 2 and 5)
  - At a stable point of operation, the response remains stable, but when stones are pushed with sufficiently large forces, the anvil may become unstable (the stone may fall to the left of point 2 or to the right of point 5.
  - A stone in an unstable area may, with a suitable input, end up in a stable area. For example, from point 1 you can get past the peak in a stable valley, but from Point 6 it is much harder to end up with a stable solution, because the stone collects so much kinetic energy that it easily slips through a stable valley into a new unstable area.



Lyapunov Stability

- State space equation $\dot{x}(t) = Ax(t) + Bu(t)$

  can lead to a general solution:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau, \quad \Phi(t) = \mathbf{e}^{\mathbf{A}t}$$

- The mode x (t) behavior in the future depends on two terms: the free response(initial values) and the forced response.
- The Lyapunov stability is the study of stability of the free response. In simple terms, it is possible to deviate from the initial state a little and see what happens when no external *u* controls are used. If the linear system is stable for one initial value, it is also stable for all other arbitrary initial values.
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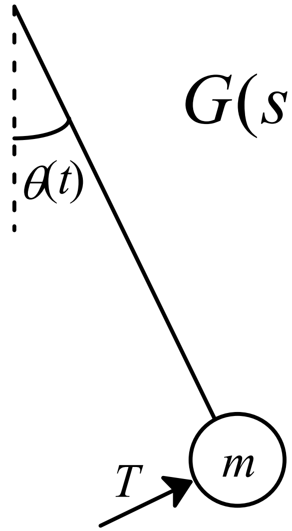
BIBO-Stability

- In the BIBO stability, on the other hand, the stability of the input / output behavior is studied and is closely related to the second term of the above-described state presentation solution. The system is BIBO (bounded input - bounded output) if any limited control u always gives a limited response y.
- For example, the integrator and the ideal pendulum (harmonic oscillator) are marginally stable (generally stable) and Lyapunov stable, but not asymptotically stable and not BIBO-stable.
- Inverted pendulum and rocket in space are unstable according to all stability criteria.
- The ideal mixer (low pass filter) and the mass block on the spring and damper are stable according to all the stability criteria outlined above.
- Asymptotic and marginal stability can be investigated by a linear system, e.g. impulse response behaviour.

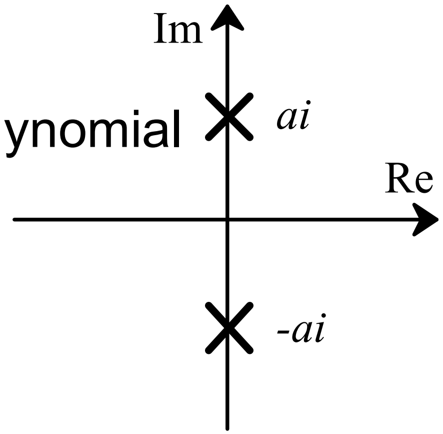
Example:

- Harmonic oscillator (Linear approximation)

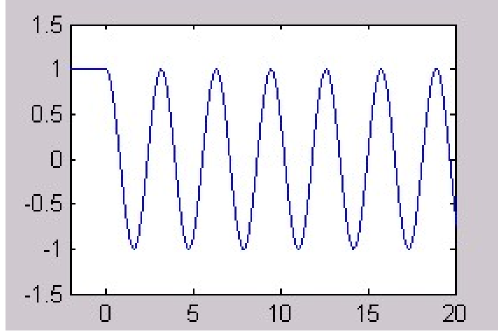


$$G(s) = \frac{a^2}{s^2 + a^2}$$

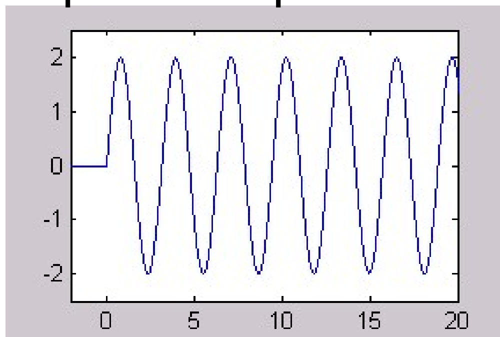
Denominator polynomial
Roots



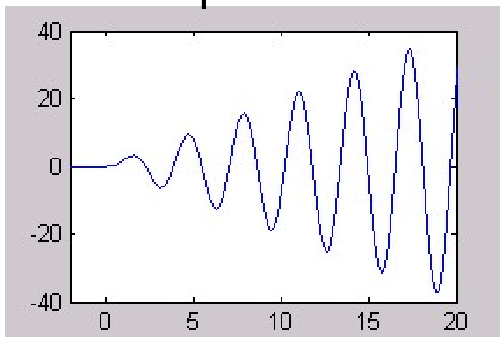
Response to initial value



Impulse response



Blue-response



- With the blue excitation, the pendulum gets into resonance and becomes unstable => the response due to the initial values is stable, but there is a limited excitation (blue signal) with which the response becomes unlimited.
- The linearized pendulum is a Lyapunov stable but not BIBO stable

Impulse response and stability

- The impulse response, i.e. the weighting function, of a system that is strictly proper (strictly proper: numerator lower-order than denominator) is the inverse Laplace transformation of the transfer function.
  - The transfer function can be transformed by a fractional break-up to a sum of fractions (assuming that all roots are real and simple at first)

$$G(s) \doteq \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-2} s^2 + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-2} s^2 + a_{n-1} s + a_n} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-2} s^2 + b_{n-1} s + b_n}{(s - s_{p1})(s - s_{p2}) \cdots (s - s_{p(n-1)})(s - s_{pn})}$$

$$= \frac{K_1}{s - s_{p1}} + \frac{K_2}{s - s_{p2}} + \cdots + \frac{K_{n-1}}{s - s_{p(n-1)}} + \frac{K_n}{s - s_{pn}}$$

$$g(t) = K_1 e^{s_{p1} t} + K_2 e^{s_{p2} t} + \cdots + K_{n-1} e^{s_{p(n-1)} t} + K_n e^{s_{pn} t}$$

  - Inverse Laplace transformation:
  - If all the denominator polynomial roots are negative, then as time approaches infinity the response approaches zero – if even one root is positive, then the corresponding sum term approaches infinity. When one sum term approaches infinity, the whole sum approaches infinite.

Impulse response and stability

- If there are complex roots among the roots, then they can be used to expand the second-degree denominator:

$$\frac{As+B}{\left(s-\mathrm{Re}\{s_{pi}\}\right)^2+\left(\mathrm{Im}\{s_{pi}\}\right)^2}, \quad \text{roots: } \mathrm{Re}\{s_{pi}\}\pm\mathrm{Im}\{s_{pi}\}\cdot i$$

- Inverse transform:

$$Ae^{\mathrm{Re}\{s_{pi}\}t}\cos\left(\mathrm{Im}\{s_{pi}\}t\right)+\left(B+A\,\mathrm{Re}\{s_{pi}\}\right)e^{\mathrm{Re}\{s_{pi}\}t}\sin\left(\mathrm{Im}\{s_{pi}\}t\right)$$

- If the real parts of the denominator polynomial are negative, then as time approaches infinity the response approaches zero. If there is even one positive real part, then the response approaches infinite.
- Let's look at the multiple roots. Multiple (Q-fold) roots in a fractional-developed form:

$$\frac{Q_q}{(s-s_{pi})^q}+\frac{Q_{q-1}}{(s-s_{pi})^{q-1}}+\cdots+\frac{Q_2}{(s-s_{pi})^2}+\frac{Q_1}{s-s_{pi}}$$

- Inverse response:

$$\frac{Q_q}{(q-1)!}t^{q-1}e^{s_{pi}t}+\frac{Q_{q-1}}{(q-2)!}t^{q-2}e^{s_{pi}t}+\cdots+Q_2te^{s_{pi}t}+Q_1e^{s_{pi}t}$$

Impulse response and stability

- If the polynomial-multiple root of the denominator is negative, then as time approaches infinity, the response to all terms approaches zero. If positive, then response approaches infinity.
- Multiple complex roots provide a response of the form:

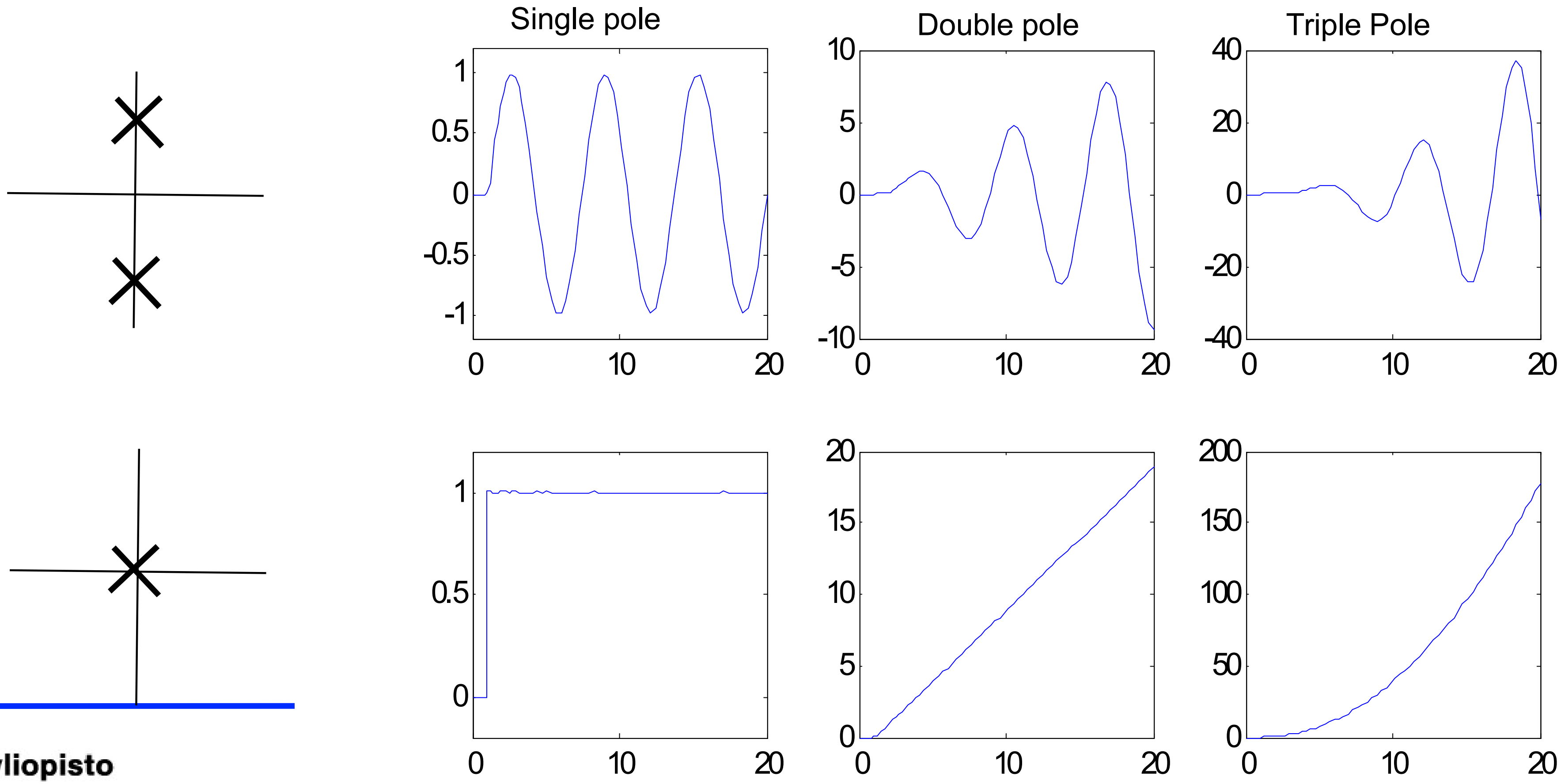
$$K_{i1} t^i e^{\mathrm{Re}\{s_{pi}\}t} \sin(\mathrm{Im}\{s_{pi}\}t) \quad \text{ja} \quad K_{i2} t^i e^{\mathrm{Re}\{s_{pi}\}t} \cos(\mathrm{Im}\{s_{pi}\}t)$$

- These terms approach zero as the time goes to infinity, if the real part of a multiple complex root is negative. If the real part is positive, then the terms approach infinity.

- In summary, an impulse response approaches zero as the time approaches infinity only if the real part of each denominator polynomial root is negative. This is the criterion of asymptotic stability. If the real part of even one root is positive, the response approaches infinity as the time approaches infinity and the system is unstable.

Marginal stability

- If all the poles of the system are on the left half-plane of the complex plane, then the system is asymptotically stable and if one or more of the poles is on the right half-plane, then the system is unstable - what if the system has no poles on the right half-plane but on the border, i.e. on the imaginary axis – then what is the impulse response?

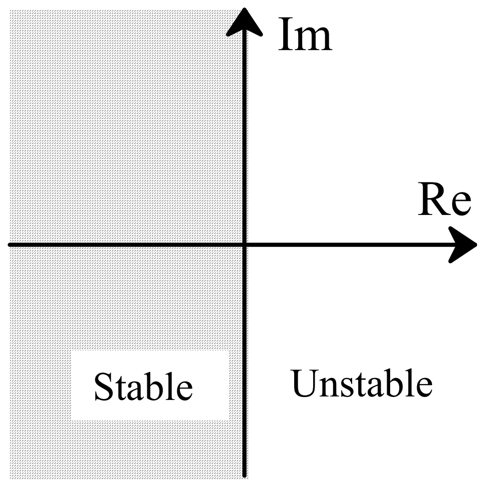


Marginal stability

- If the system has single poles on the imaginary axis and no poles on the right side of the plane, it is marginally stable (stable but not asymptotically stable).
- If the system has multiple poles on the imaginary axis, it is unstable according to all stability criteria.
- If the system has one pole on the imaginary axis (at the origin of the complex plane), it is marginally stable, not asymptotically stable nor BIBO-stable.

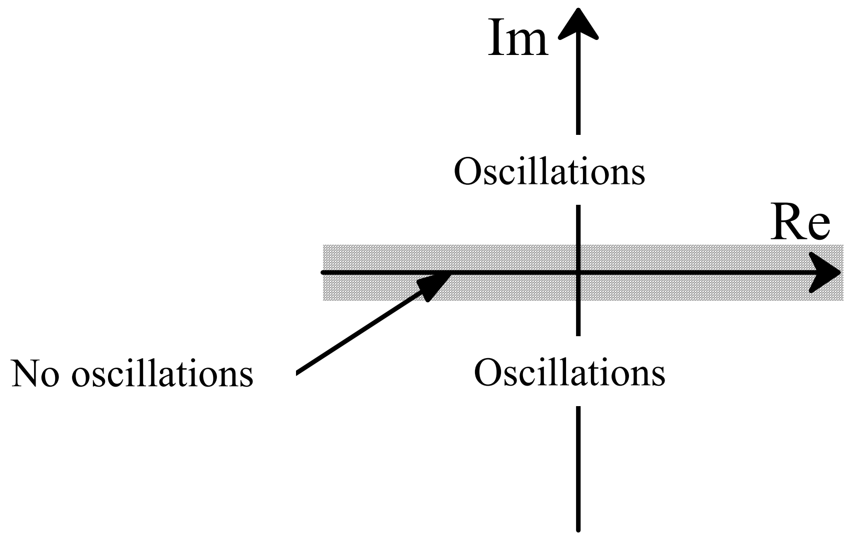
Stability

- The system is asymptotically stable, if all its poles are on the left side of a complex plane.
- The system is unstable if it has one or more poles at the right side of a complex plane or has multiple poles on the Imaginary axis
- The system is marginally stable if it has one or more simple poles on the imaginary axis and not a single pole on the right half plane.
- Asymptotic stability implies BIBO stability. BIBO stability implies asymptotic stability, if the system is both *reachable* and *observable* (these concepts are discussed later).



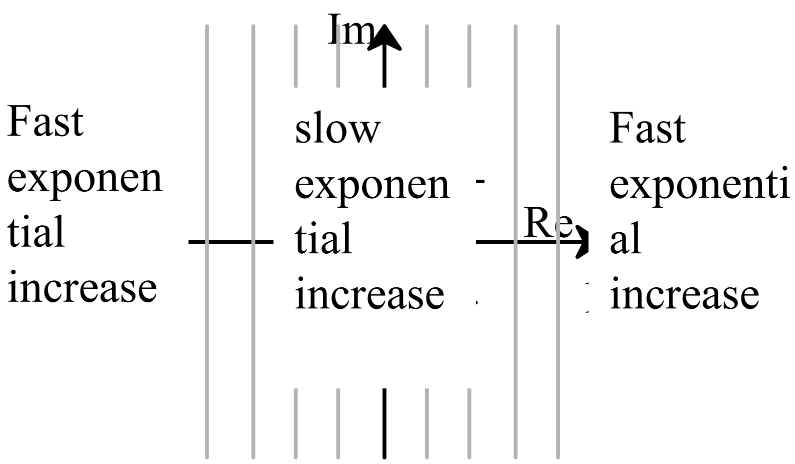
Oscillations

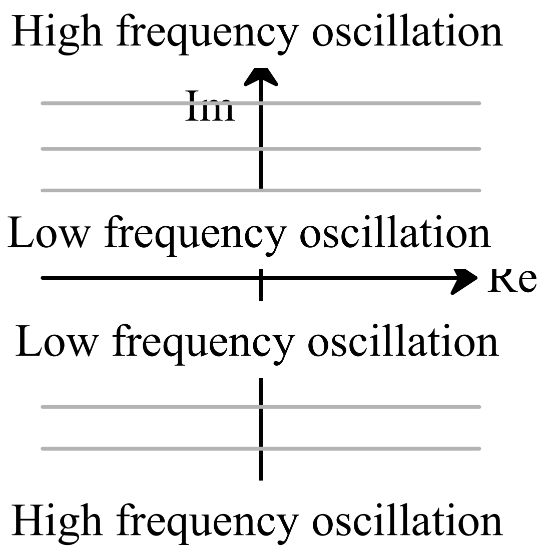
- For Laplace conversion pairs, it is known that if the transfer function has complex roots in the denominator, then the inverse Laplace transformation gives sine and cosine functions. That means that the response *oscillates.*
- Just as in the case of stability, if one part of the term oscillates, the whole system oscillates.
- The system response does not oscillate if the poles are on the real axis. The response oscillates if even one polar pair is truly complex (not located on the real axis).

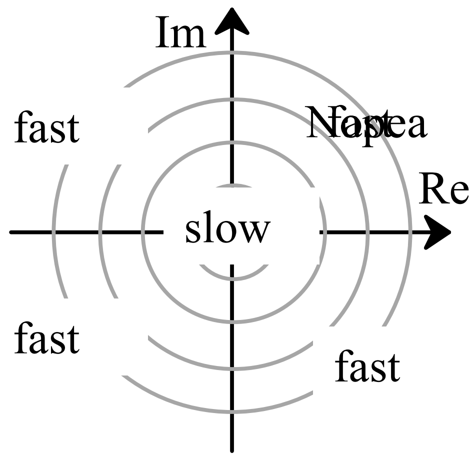


Speed

- The distance from the imaginary axis illustrates the exponential behavior (the farther from the imaginary axis the faster the system reaches its final value or escapes to infinity).
- The distance from the real axis is illustrated by the frequency of vibration (the greater the distance from the real axis, the greater is the frequency).
- The system is faster, the farther its poles are from the origin



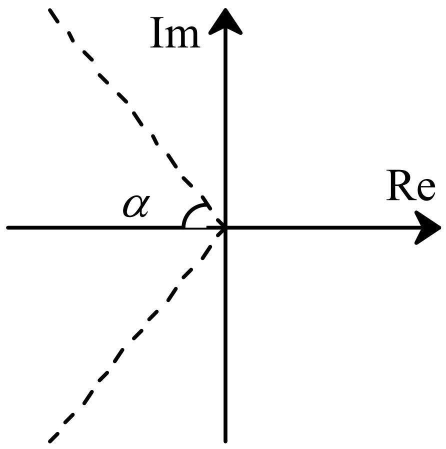




Damping Ratio of oscillation

- The damping ratio of the system illustrates the oscillation damping capability.
- The damping ratio is calculated from the cosine of the angle between the complex pole and the negative real axis – this only applies when the angle exists – that is, a genuinely complex pole.
- If the damping ratio is 1, the system is critically damped (no oscillation).
- If the damping ratio is between 1 and 0, the system is underdamped(a genuinely complex polar pair on the left side).
- If the damping ratio is zero, the system is a harmonic oscillator (poles on the imaginary axis).
- If the damping ratio is negative, the system is unstable (poles on the right half plane).

$$\varsigma = \cos(\alpha)$$



$$G(s) = \frac{k\omega_0^2}{s^2 + 2\varsigma\omega_0 s + \omega_0^2}$$

Second order system

ζ is the *damping ratio*

ω_0 is the *natural frequency*

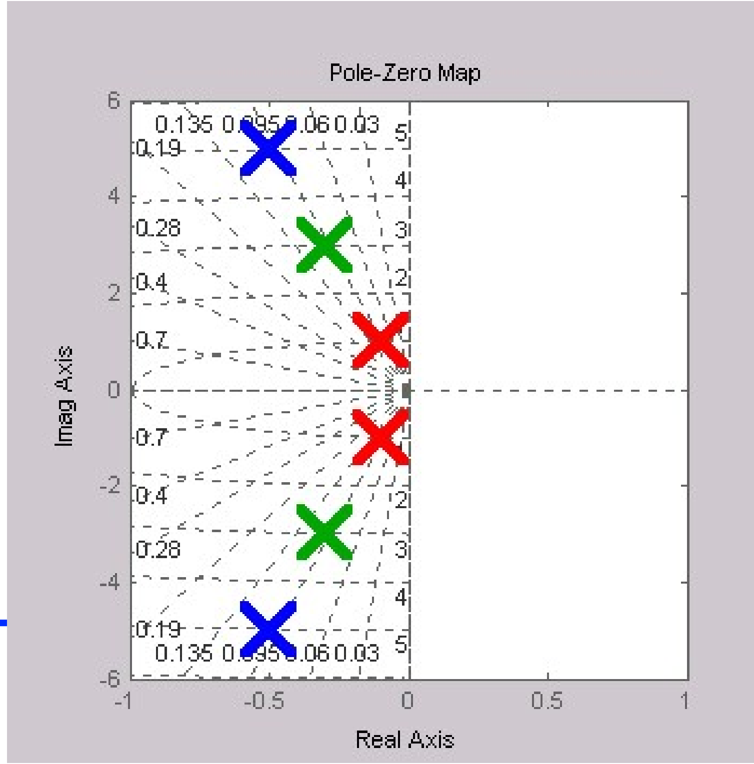
Example: damping ratio and responses

- Examining three different processes

$$\begin{cases} G_1(s) = \dfrac{5}{s^2+s+25} & \Rightarrow \quad \zeta = 0.1, \omega_n = 5 \quad \Leftrightarrow \quad s_{p1,2} = -0.5 \pm 4.97i \\ G_2(s) = \dfrac{3}{s^2+0.6s+9} & \Rightarrow \quad \zeta = 0.1, \omega_n = 3 \quad \Leftrightarrow \quad s_{p1,2} = -0.3 \pm 2.98i \\ G_3(s) = \dfrac{1}{s^2+0.2s+1} & \Rightarrow \quad \zeta = 0.1, \omega_n = 1 \quad \Leftrightarrow \quad s_{p1,2} = -0.1 \pm 0.99i \end{cases}$$

- All have the same damping ratio, but different distance from the origin.
- Plotting the pole zero map in Matlab:

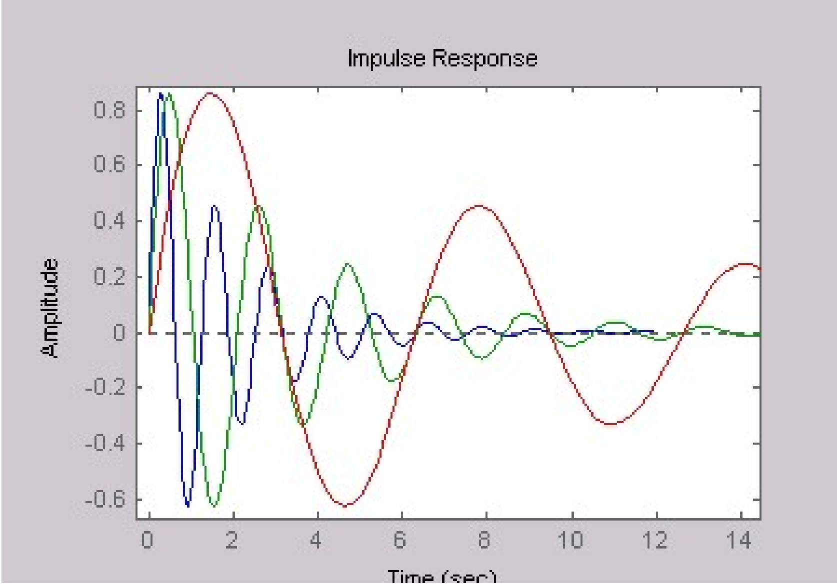
```
sys1=tf(5,[1 1 25])
sys2=tf(3,[1 0.6 9])
sys3=tf(1,[1 0.2 1])
pzmap(sys1,sys2,sys3)
sgrid
```



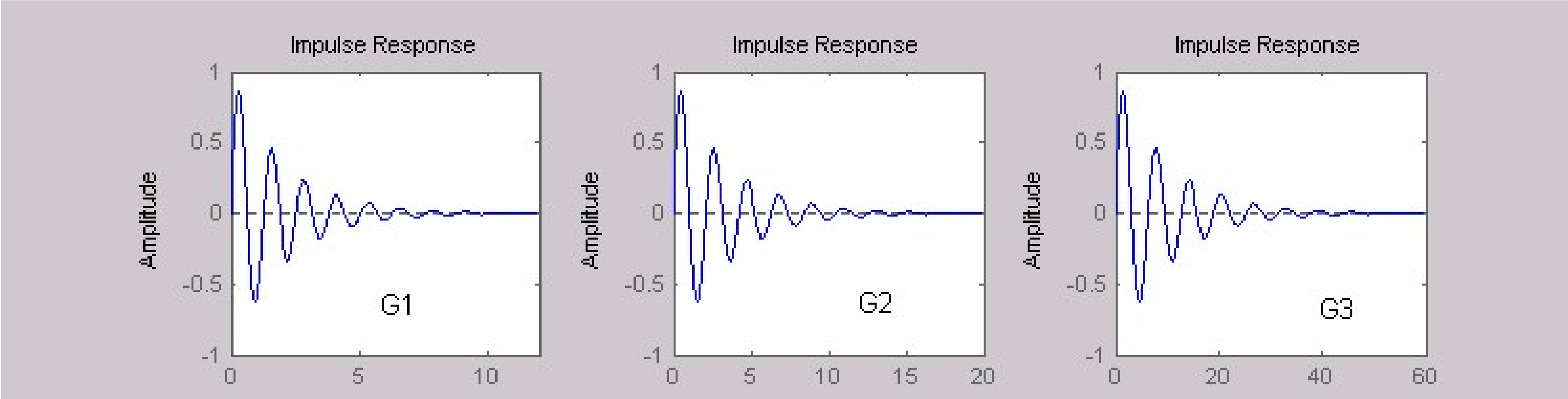
Example: damping ratio and responses

- The response:

```
impulse(sys1)
impulse(sys2)
impulse(sys3)
```



- The system has different speeds, but the damping of oscillations in each system is the same .
- If each impulse response were scaled in time, the responses would be identical.



Dominant poles

- Consider the system: $$G(s) = \frac{1000}{(s+1)(s+10)(s+100)}$$
- The step response is

$$y(t) = 1 - 1.1223e^{-t} + 0.1235e^{-10t} - 0.0011e^{-100t}$$

- The farther the pole is from the imaginary axis, the lower is its coefficient in the response expression.
- The poles and pole pairs that dominate the stable behavior are the ones closest to the Imaginary axis – they are the *dominant poles*. The same is true for zeros and zero pairs.
- However, unstable behavior is always dominant.

Zeros

- If the poles affect stability, oscillation and speed, then what do the zeros do?
- Take an example: a process with two stable poles (points-2 and-3) and one zero.

$$G(s) = \frac{(\tau_3 s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{(\tau_3 s + 1)}{(\frac{1}{3} s + 1)(\frac{1}{2} s + 1)}$$

- Calculating step response:

$$Y(s) = \frac{(\tau_3 s + 1)}{(\frac{1}{3} s + 1)(\frac{1}{2} s + 1)s} = \frac{1}{s} + \frac{2 - 6\tau_3}{s + 3} + \frac{6\tau_3 - 3}{s + 2}$$

$$\Rightarrow \quad y(t) = 1 + (2 - 6\tau_3)e^{-3t} + (6\tau_3 - 3)e^{-2t}$$

- Zeros affect the numerators of different factors. Let us test different zero values and examine how the step response changes.

τ_3 takes values $-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$

(Corresponding zeros are: : $\frac{1}{2}, 1, 2,$ No zero $, -2, -1, -\frac{1}{2}$)

Zeros

- The zeros affect the initial behavior of the responses.
- If the system has one or more zeros on the right side of the complex plane, the system is *non-minimum phase* and, conversely, if all its zeros are on the left side (and there is no delay), then it is *minimum-phase*.