ELEC-C8201 Control and Automation

Lecture 4: Routh-Hurwitz stability test

Stability tests

- If the roots of the system are known (the denominator polynomial zero points), then stability is easy to observe.
  - Roots can be determined from numerical polynomial by iterative calculation routines (by Matlab commands such as **`eig`**, **`roots`** and **`pole`**).
    - E.g. Polynomial $s^3 + 2s^2 + 4s + 10$

```
roots([1 2 4 10])
ans =         -2.2236
               0.1118 + 2.1177i
               0.1118 - 2.1177i
```

  - If one of the polynomial coefficients is zero or negative, then the polynomial has at least one root on the imaginary axis or the right half plane.
  - If the polynomial contains symbolic parameters and you want to determine at which parameter values the system is stable, then the root solution numerically will no longer be successful. You can then use the Routh's chart.
  - The method is given in the following without proof (which would need a reasonable study of polynomial algebra).

Routh's Chart

- Consider polynomial $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ to generate the Routh's chart::

$$
\begin{array}{c|cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & a_8 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & a_9 & \cdots \\
\hline
s^{n-2} & b_0 & b_2 & b_4 & b_6 & \cdots & \\
s^{n-3} & b_1 & b_3 & b_5 & b_7 & \cdots & \\
\hline
s^{n-4} & c_0 & c_2 & c_4 & \cdots & & \\
s^{n-5} & c_1 & c_3 & c_5 & \cdots & & \\
\hline
\vdots & \vdots & & & & & \\
\hline
s^1 & z_0 & & & & & \\
s^0 & z_1 & & & & &
\end{array}
$$

$$
b_0 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, \quad b_2 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}, \quad b_4 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}, \quad \cdots
$$

$$
b_1 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_3 \\ b_0 & b_2 \end{vmatrix}, \quad b_3 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_5 \\ b_0 & b_4 \end{vmatrix}, \quad b_5 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_7 \\ b_0 & b_6 \end{vmatrix}, \quad \cdots
$$

$$
c_0 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_2 \\ b_1 & b_3 \end{vmatrix}, \quad c_2 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_4 \\ b_1 & b_5 \end{vmatrix}, \quad c_4 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_6 \\ b_1 & b_7 \end{vmatrix}, \quad \cdots
$$

$$\vdots$$

$$z_1 = a_n$$

Routh's Chart

- The number of sign changes in the first column of the Routh's chart is also the number of roots at the right side of the complex plane.
- If the typical polynomial of the system is placed in the Routh diagram, the system is stable if there is no sign change in the first column.
- If there is a zero in the first column of the chart, it is replaced by the small positive number e in the diagram and forming diagram is continued. The final chart can be used to calculate the sign changes by examining the limits of the terms that depend on $e \rightarrow 0$.
- If the chart consists of a whole row of zeros, then the original polynomial is divisible by another polynomial formed by the coefficients above the zero line.

Examples: Routh's chart

- Polynomials:
-

$$s^3+2s^2+4s+10$$

s^3	1	4
s^2	2	10
s^1	-1	
s^0	10	

Two sign changes 2→-1
and -1→10
So two roots on the right
half-plane (RHP).

$$s^4+4s^3+6s^2+4s+2$$

s^4	1	6	2
s^3	4	4	
s^2	5	2	
s^1	$12/5$		
s^0	2		

No sign changes in first column
So no roots on the right
half-plane

Examples: Routh's chart

- Polynomial: $s^3 + s^2 + 2s + 2$

s^3	1	2
s^2	1	2
s^1	0	0
s^0		

A zero row is obtained, resulting in a higher line polynomial $s^2 + 2$ with which the original polynomial is divisible.

Another way: Take the derivative of the auxiliary polynomial and continue.

$$\frac{d}{ds}(s^2 + 2) = 2s$$

s^3	1	2
s^2	1	2
s^1	2	0
s^0	2	

No character changes, so no roots on the right side of the plane

Examples: Routh's chart

- Polynomial: $s^4 + 3s^3 + 4s^2 + 12s + 12$

s^4	1	4	12
s^3	3	12	
s^2	$0 \to \varepsilon$	12	
s^1	$(12\varepsilon - 36)/\varepsilon$		
s^0	12		

The first column becomes zero, replace it with a low positive number ϵ and continue

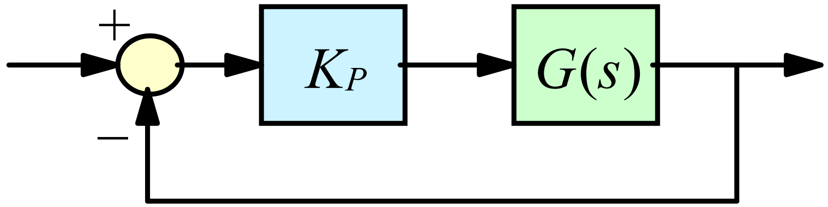
$$\lim_{\varepsilon \to 0} \left\{ \frac{12\varepsilon - 36}{\varepsilon} \right\} = -\infty$$

s^4	1	4	12
s^3	3	12	
s^2	0	12	
s^1	$-\infty$		
s^0	12		

Two sign changes $0 \to -\infty$ ja $-\infty \to 12$
$\Rightarrow$ Two roots on the right side of the plane

Examples: Routh's chart

- System with the transfer function is controlled with the P controller. $G(s) = \dfrac{1}{s^3 + 4s^2 + s - 6}$
- For what values of K_P is the system stable?



$$G_{TOT}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{\dfrac{K_P}{s^3 + 4s^2 + s - 6}}{1 + \dfrac{K_P}{s^3 + 4s^2 + s - 6}} = \frac{K_P}{s^3 + 4s^2 + s + (K_P - 6)}$$

$$\begin{array}{c|c:c} s^3 & 1 & 1 \\ s^2 & 4 & K_P - 6 \\ \hdashline s^1 & \dfrac{10 - K_P}{4} & \\ s^0 & K_P - 6 & \end{array}$$

Stable if,

$$\frac{10 - K_P}{4} \geq 0 \text{ ja } K_P - 6 \geq 0$$

$$\Rightarrow 6 \leq K_P \leq 10$$

State space poles and zeros

- Earlier the conversion between transfer function and state space representation was derived.

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

  - The inverse of the matrix is calculated by dividing the adjoint matrix with the determinant.

$$\mathbf{G}(s) = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}}{\det(s\mathbf{I} - \mathbf{A})} + \mathbf{D} = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

Characteristic polynomial:

System poles equation

System zeros equation

$$\begin{cases} \det(s\mathbf{I} - \mathbf{A}) = 0 \\ \det(s\mathbf{I} - \mathbf{A}) = 0 \\ \mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A}) = 0 \end{cases}$$

- Note that the equations apply to multivariable (MIMO) systems as well.

Example

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

- Determine system poles and zeros. Characteristic equation:

$$\det\left(s\mathbf{I} - \mathbf{A}\right) = \det\begin{bmatrix} s+1 & -2 \\ 0 & s-3 \end{bmatrix} = (s+1)(s-3) = 0$$

$$\mathbf{C}\,\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{B} + \mathbf{D}\det\left(s\mathbf{I} - \mathbf{A}\right) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathrm{adj}\begin{bmatrix} s+1 & -2 \\ 0 & s-3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-3 & 2 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2(s+3) & s-3 \end{bmatrix}$$

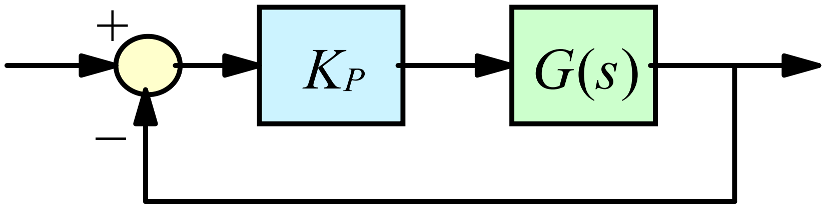
$$\Rightarrow \quad \mathbf{G}(s) = \begin{bmatrix} \frac{2(s+3)}{(s+1)(s-3)} & \frac{s-3}{(s+1)(s-3)} \end{bmatrix} = \begin{bmatrix} \frac{2(s+3)}{(s+1)(s-3)} & \frac{1}{s+1} \end{bmatrix}$$

- The transfer function has two poles (-1 ja 3) and one Zero (-3)

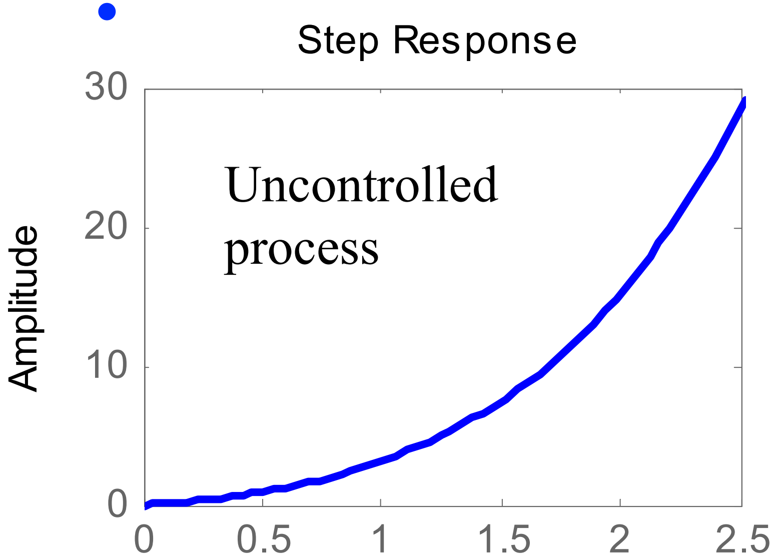
Example: Behavior of a given system

- Examine the behavior of the prescribed system with the various tuning controls (P-control).

$$G(s) = \frac{s+2}{s^2 - s} = \frac{s+2}{s(s-1)}$$



- The uncontrolled process is unstable
-



Regulated system

$$G_{TOT}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{K_P(s+2)}{s^2 - s + K_P(s+2)}$$

$$= \frac{K_P(s+2)}{s^2 + (K_P - 1)s + 2K_P}$$

Example: Behavior of the given system

- Characteristic equation of the given system $s^2 + (K_P - 1)s + 2K_P = 0$
- System poles (Quadratic equation solution):

$$s_{p,1,2} = -\frac{K_P - 1}{2} \pm \frac{\sqrt{K_P^2 - 10K_P + 1}}{2}$$

- The system is not oscillating when the poles are
- $$K_P^2 - 10K_P + 1 \geq 0 \quad \Rightarrow \quad \left(K_P - 5 + 2\sqrt{6}\right)\left(K_P - 5 - 2\sqrt{6}\right) \geq 0$$

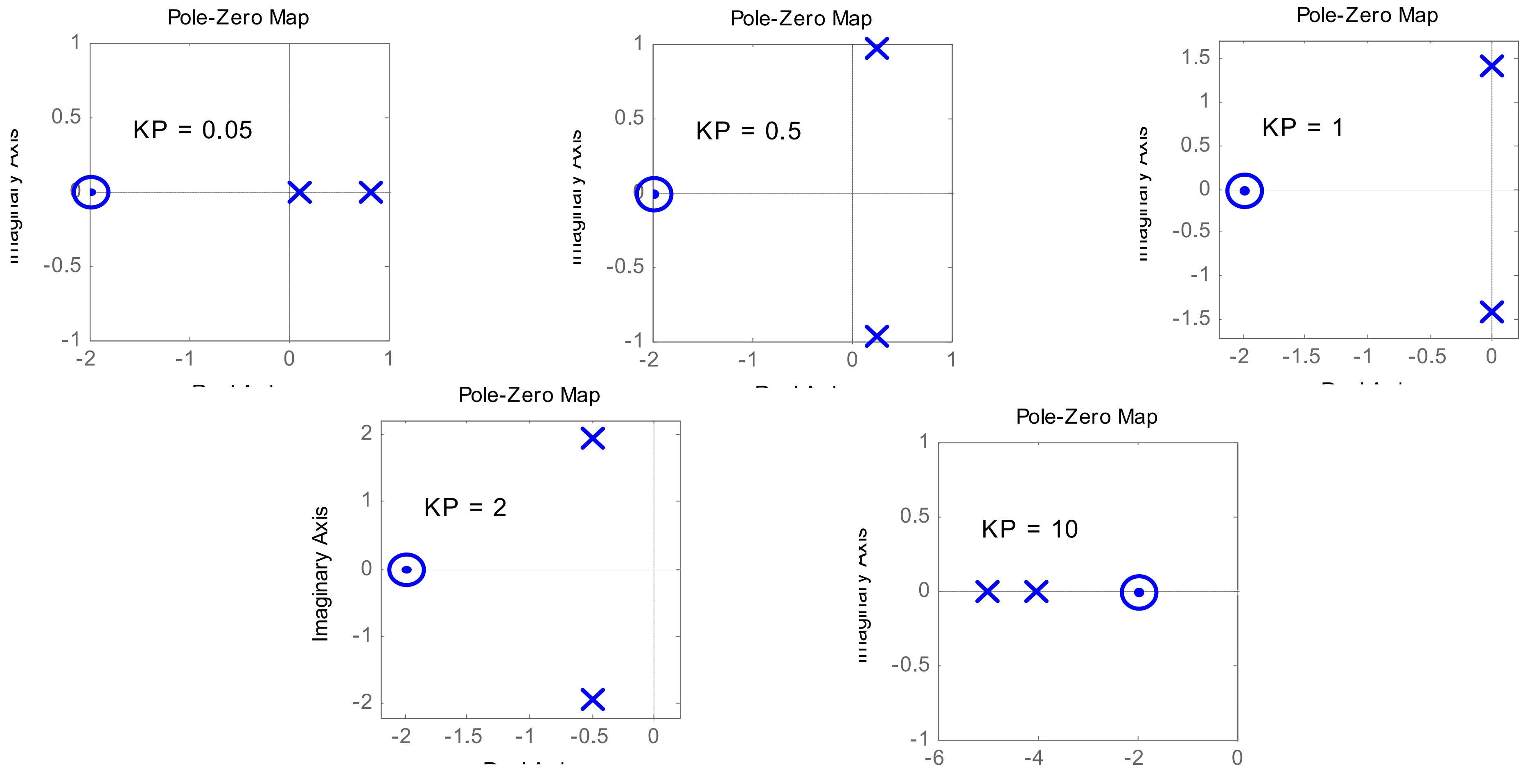
$$\Rightarrow \quad K_P \leq 5 - 2\sqrt{6} \approx 0.1010 \quad \text{or} \quad K_P \geq 5 + 2\sqrt{6} \approx 9.8990$$

- The system is stable when the poles are on the left half plane

$$K_P \geq 0 \quad \text{ja} \quad K_P - 1 \geq 0 \quad \Rightarrow \quad K_P \geq 1$$

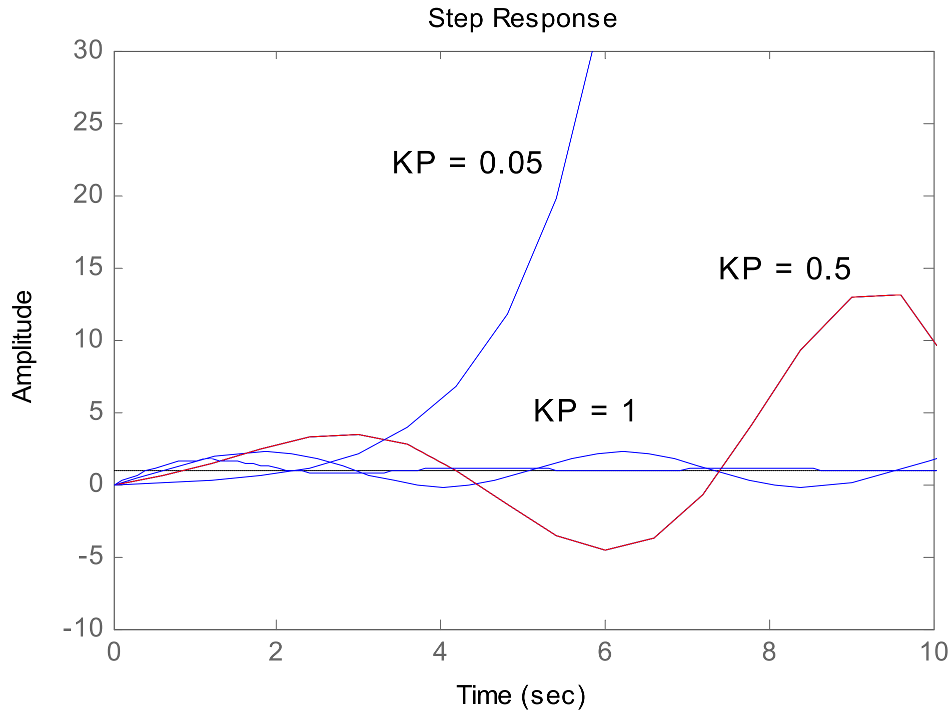
Example: Behavior of the given system

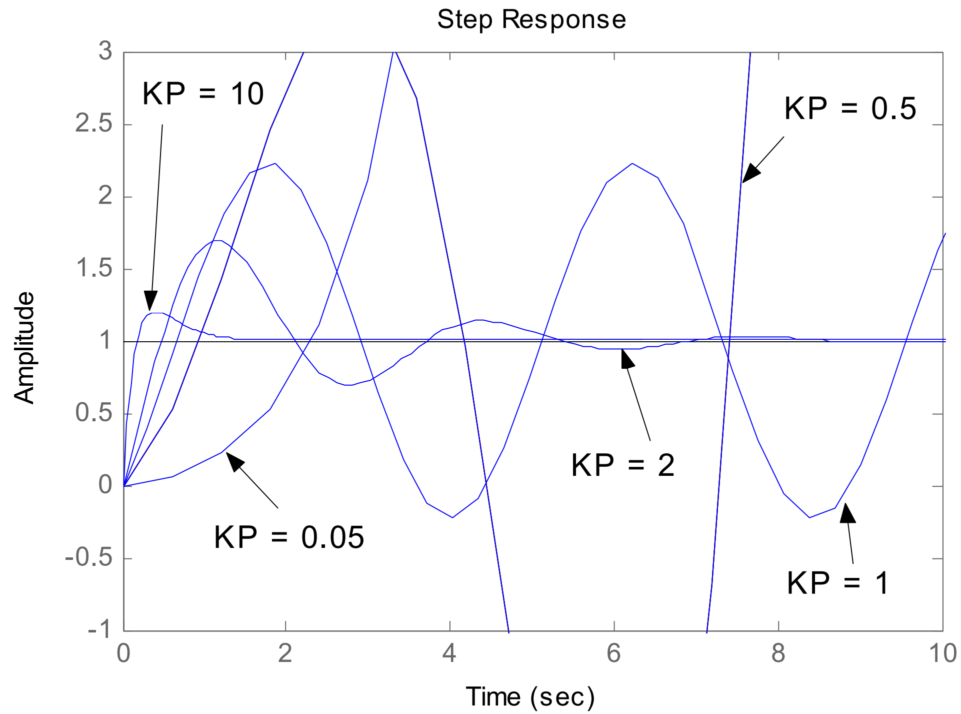
- Determine the poles and plot the pole zero maps of the given system with different K_P values



Example: Behavior of the given system

- The corresponding step responses are:





Example: Behavior of the given system

- So, a controlled system can be

$$\begin{cases} K_P \le 5-2\sqrt{6}, & \text{Response is non-oscillatory and unstable} \\ 5-2\sqrt{6} \le K_P < 1, & \text{Response has oscillations with growing amplitude and unstable} \\ K_P = 1, & \text{Response has harmonic oscillations} \\ 1 < K_P < 5+2\sqrt{6}, & \text{Response has oscillations but is stable} \\ K_P = 5+2\sqrt{6}, & \text{Response is critically stable} \\ K_P > 5+2\sqrt{6}, & \text{Response has no oscillations and is stable} \end{cases}$$

Common transfer function templates: 1st order system

- 1. First order dynamics
  - Differential equation and transfer function:
    - *K is the gain*
    - τ is the system time constant

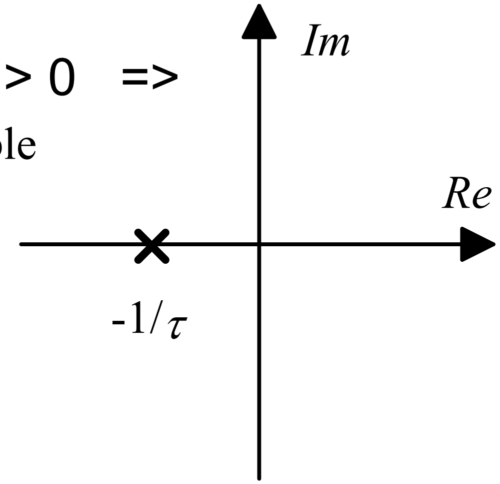
$$\tau \dot{y}(t) + y(t) = Ku(t) \qquad G(s) = \frac{K}{\tau s + 1}$$

Impulse response $\quad y(t) = \frac{K}{\tau} e^{-\frac{t}{\tau}}$

Step response $\quad y(t) = K\left(1 - e^{-\frac{t}{\tau}}\right)$

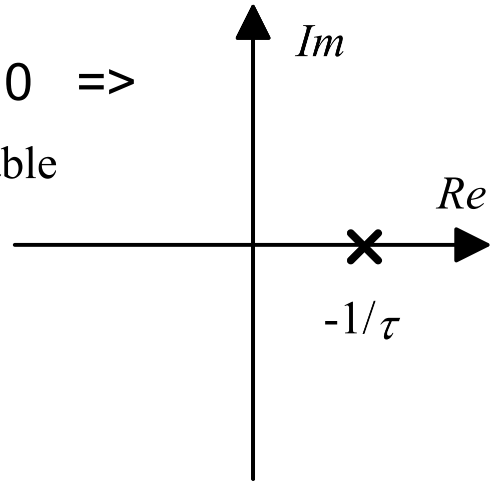
$\tau > 0$ =>

Stable



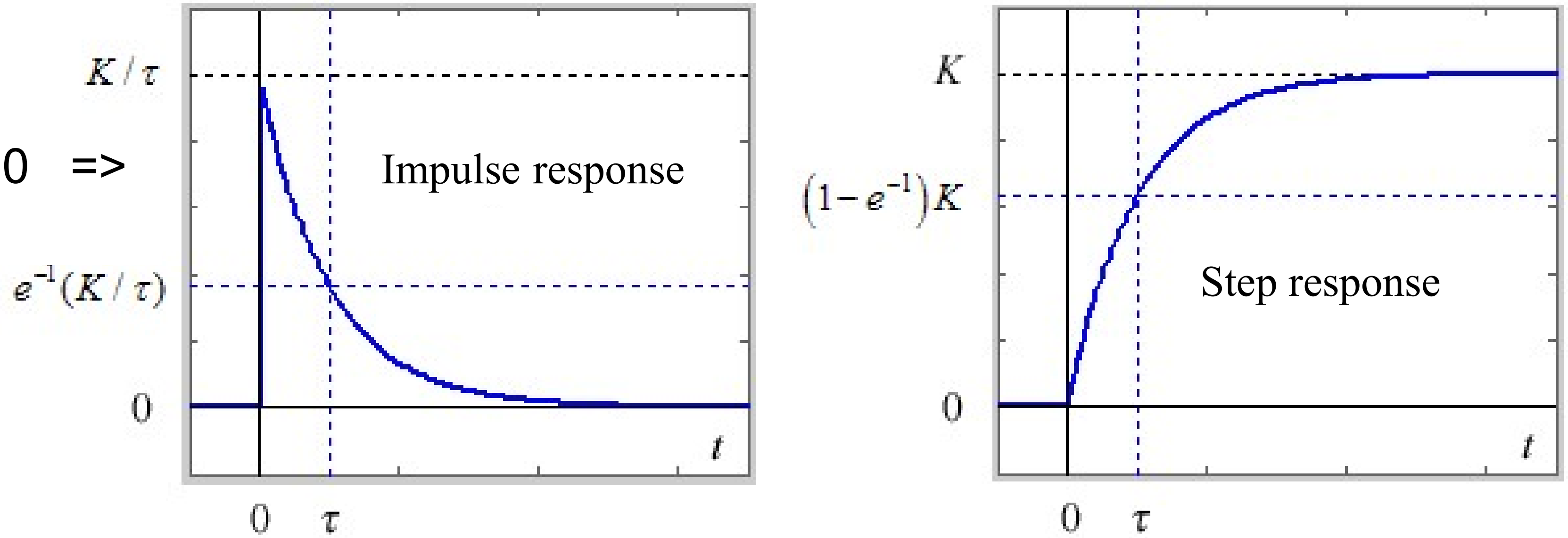
$\tau < 0$ =>

Unstable

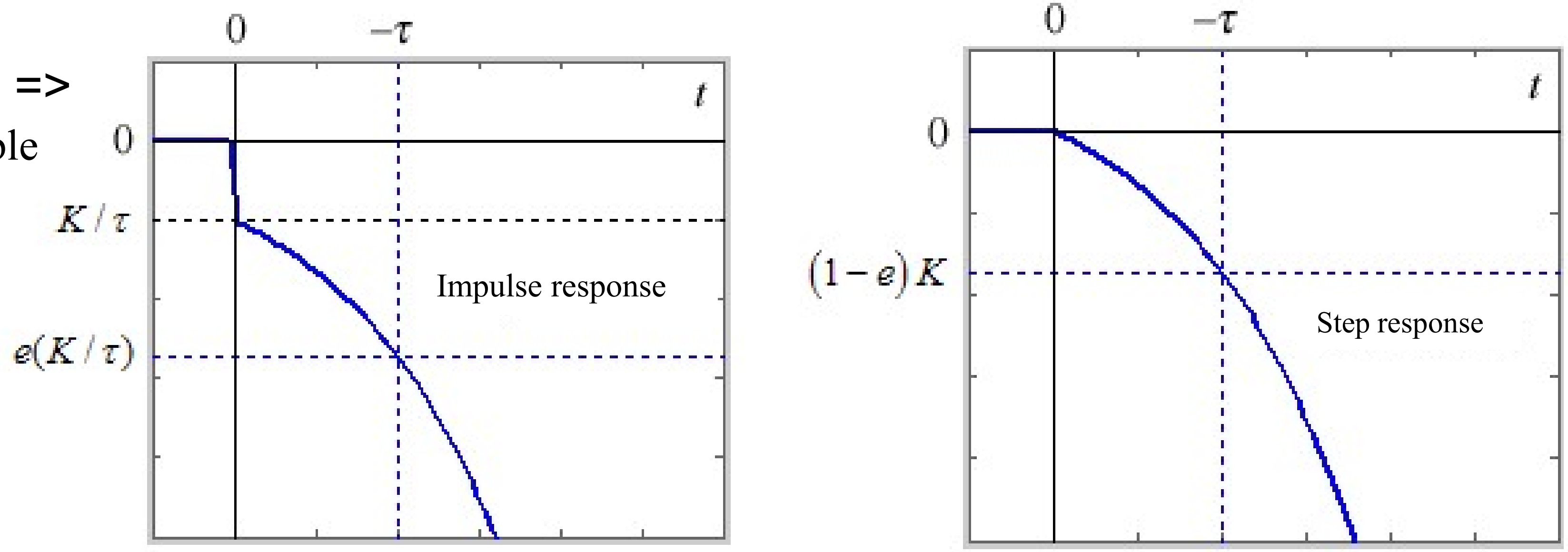


Common transfer function templates: 1st order system

$\tau > 0$ =>
Stable



$\tau < 0$ =>
Unstable



Common transfer function templates: 2nd order system

- 2nd order oscillation dynamics (complex poles)
  - Differential equation and transfer function:
    - *K is the system gain*
    - ω_n is system natural angular frequency (ω_n > 0)
    - ζ is the damping ratio of the system (-1 > ζ > 1)

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = K\omega_n^2 u(t)$$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Impulse response :

$$y(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right)$$

Step response :

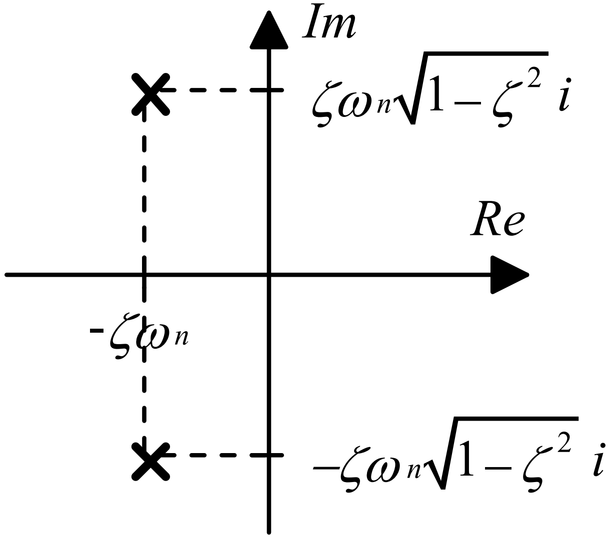
$$y(t) = K\left(1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right)\right)$$

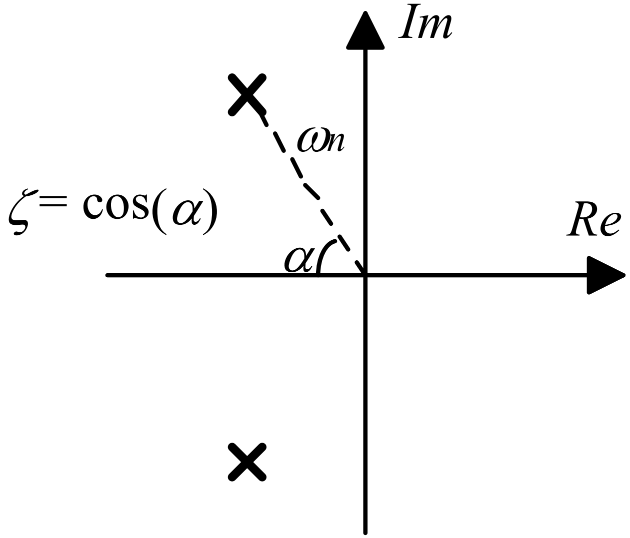
$$= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \cos^{-1}(\zeta)\right)\right)$$

Common transfer function templates: 2nd order system

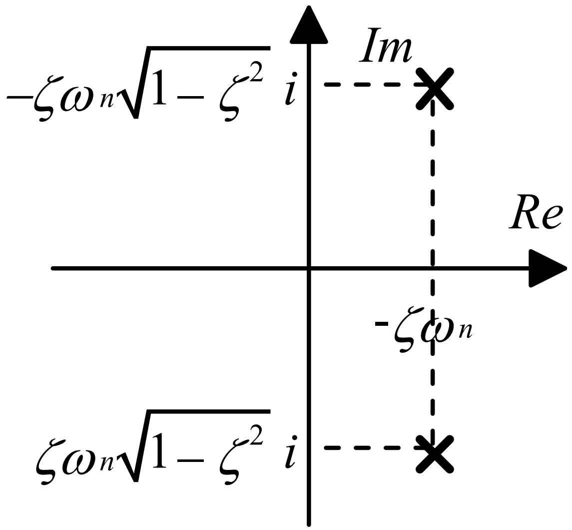
poles:

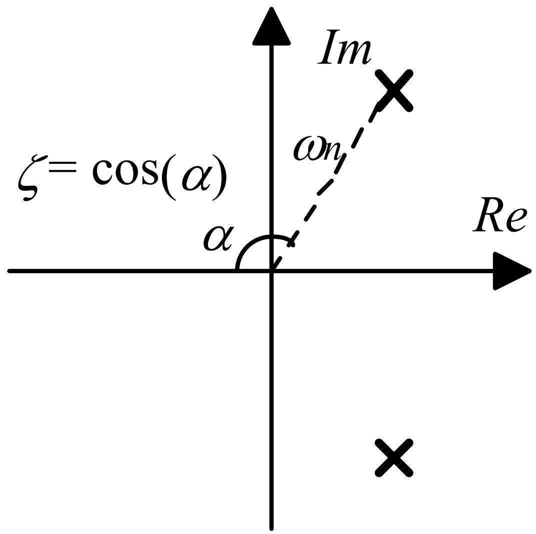
$$\begin{cases} s_{P,1} = -\omega_n\zeta + \omega_n\sqrt{1-\zeta^2}\,i \\ s_{P,2} = -\omega_n\zeta - \omega_n\sqrt{1-\zeta^2}\,i \end{cases}$$





$0 > \zeta > 1$ =>
Stable

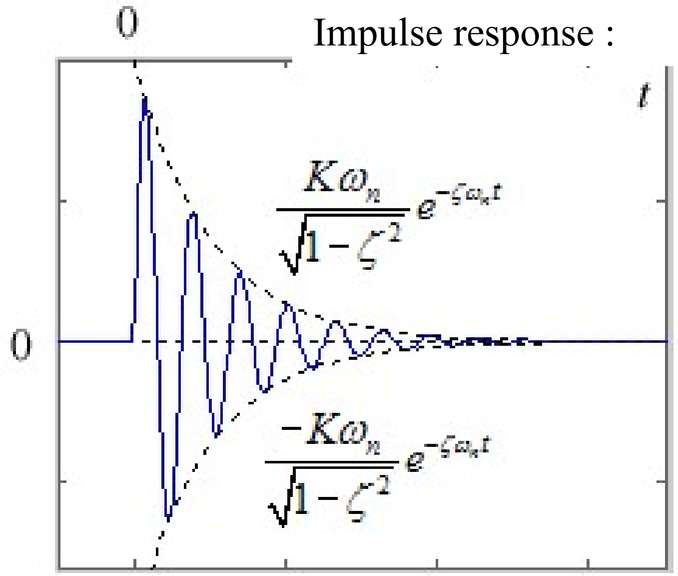


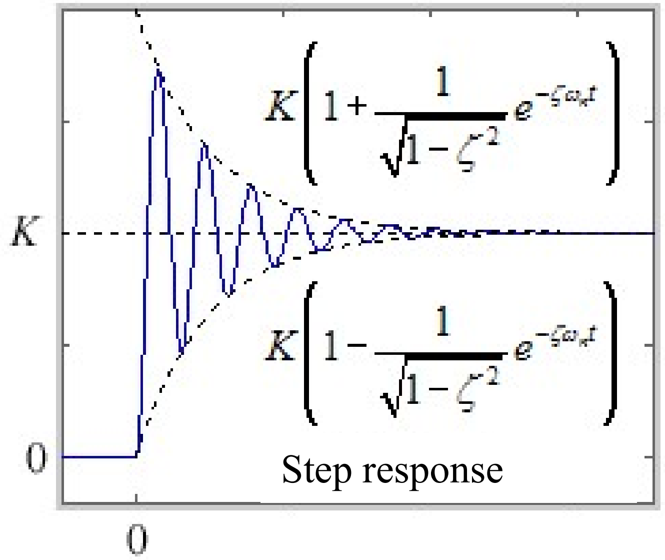


$-1 > \zeta > 0$ =>
Unstable

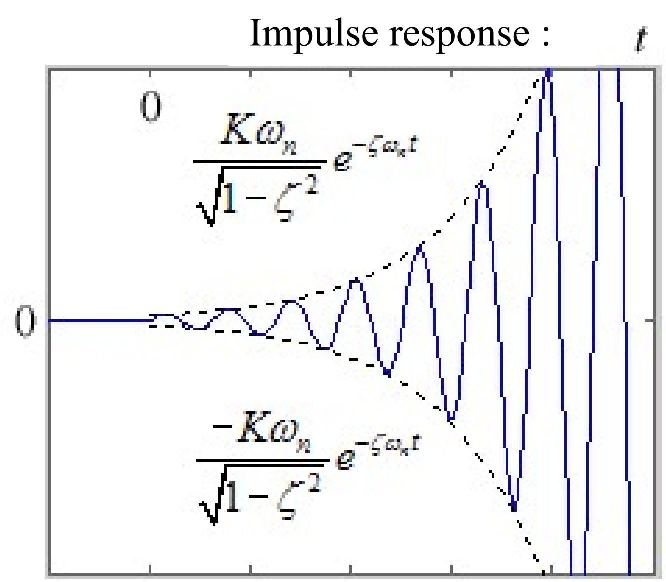
Common transfer function templates: 2nd order system

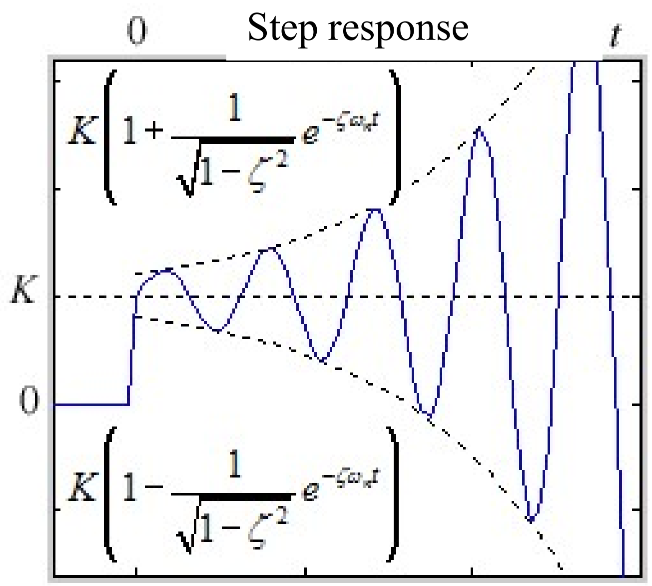
$0 > \zeta > 1$ =>
Stable





$-1 > \zeta > 0$ =>
Unstable





Common transfer function templates: 2nd order system

- The oscillation dynamics of 2nd order system (real poles)
  - Differential equation and transfer function:
    - K is system control gain
    - τ_1 and τ_2 are system time constants ($\tau_1 \neq \tau_2$)

$$\tau_1\tau_2\ddot{y}(t)+(\tau_1+\tau_2)\dot{y}(t)+y(t)=Ku(t)$$

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Impulse response : $y(t)=\frac{K}{\tau_2-\tau_1}\left(e^{-\frac{t}{\tau_2}}-e^{-\frac{t}{\tau_1}}\right)$

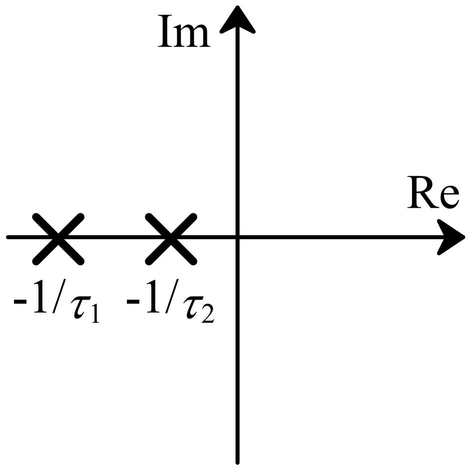
Step response : $y(t)=K\left(1-\frac{1}{\tau_2-\tau_1}\left(\tau_2 e^{-\frac{t}{\tau_2}}-\tau_1 e^{-\frac{t}{\tau_1}}\right)\right)$

Common transfer function templates: 2nd order system

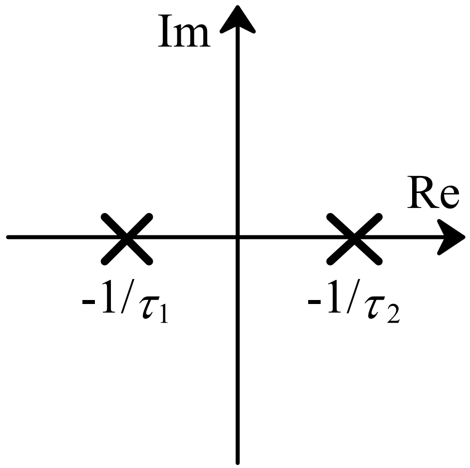
poles:

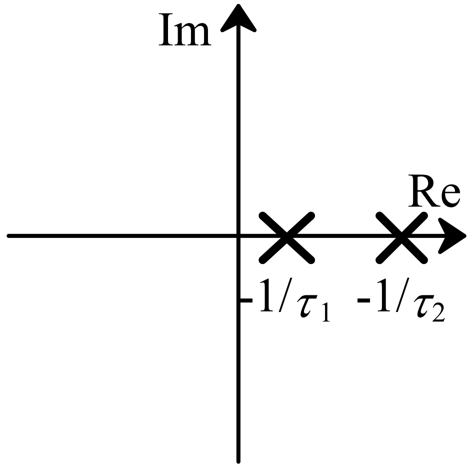
$$\begin{cases} s_{P,1} = -\dfrac{1}{\tau_1} \\ s_{P,2} = -\dfrac{1}{\tau_2} \end{cases}$$

$\tau_1, \ \tau_2 > 0$
=> stable



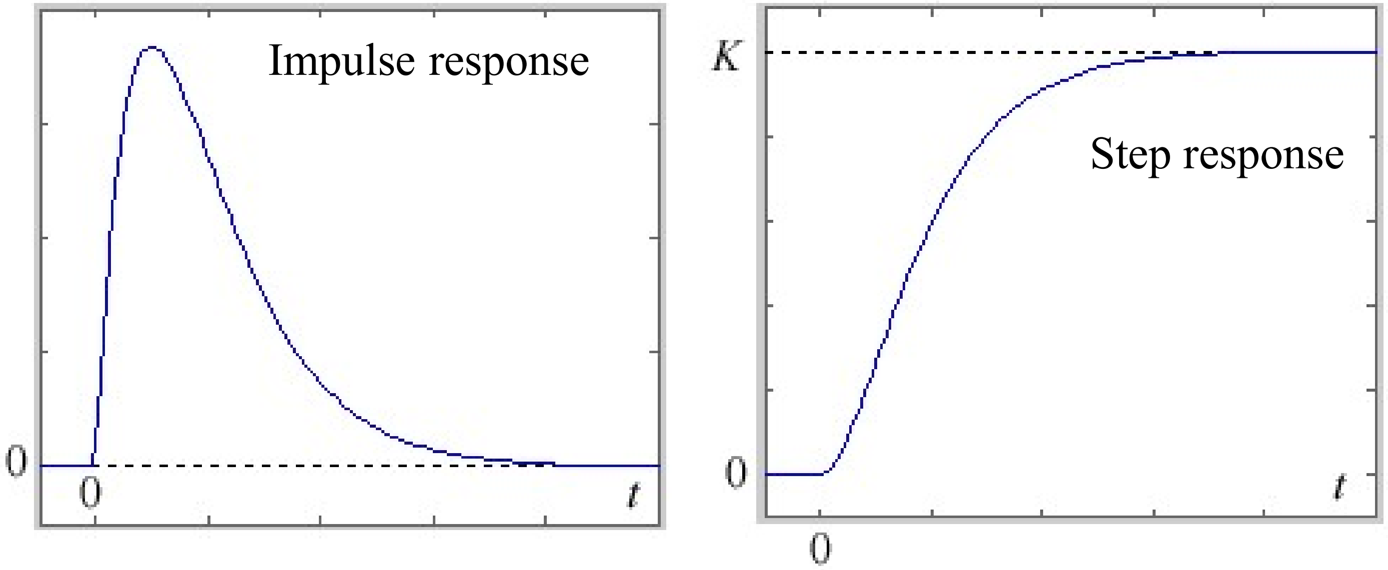
$\tau_1 < 0$ tai $\tau_2 < 0$
=> Unstable



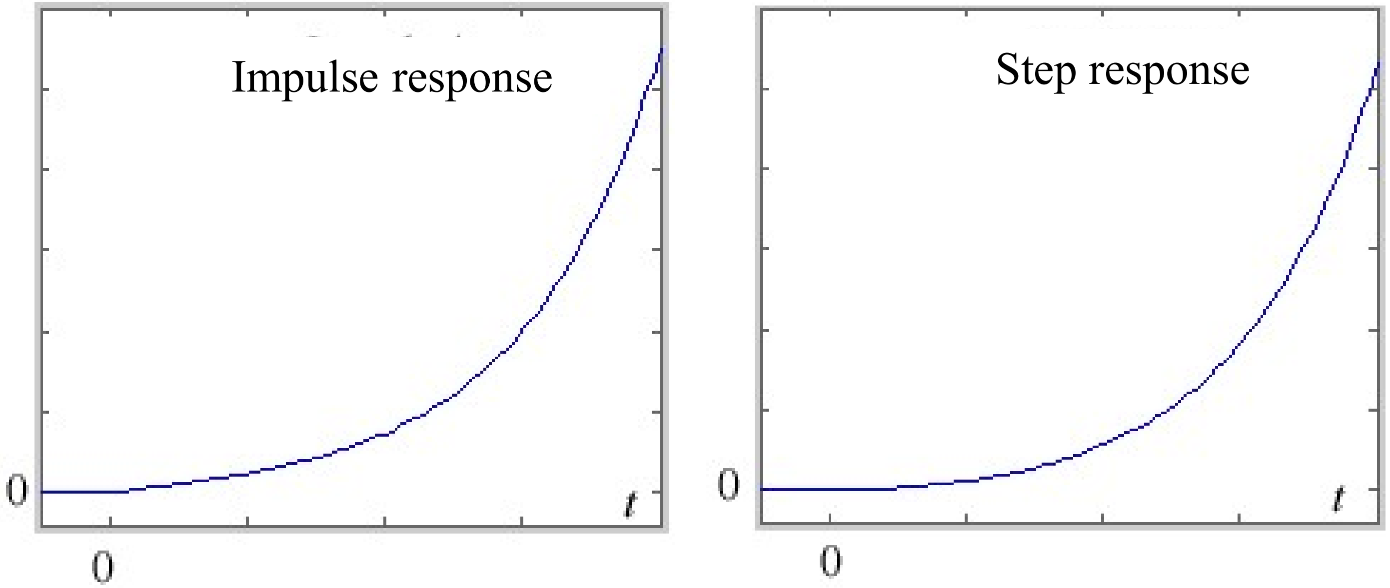


Common transfer function templates: 2nd order system

$\tau_1, \tau_2 > 0$
=> stable



$\tau_1 < 0$ or $\tau_2 < 0$
=> unstable



Higher-order Models

- Higher-order models can be formed from simple 1st and 2nd order models
- It is known that the response is a weighted sum of all elements (poles and zeros near the vertical axis dominate the behavior of the system)
  - Examining the oscillating system of the 3rd order:

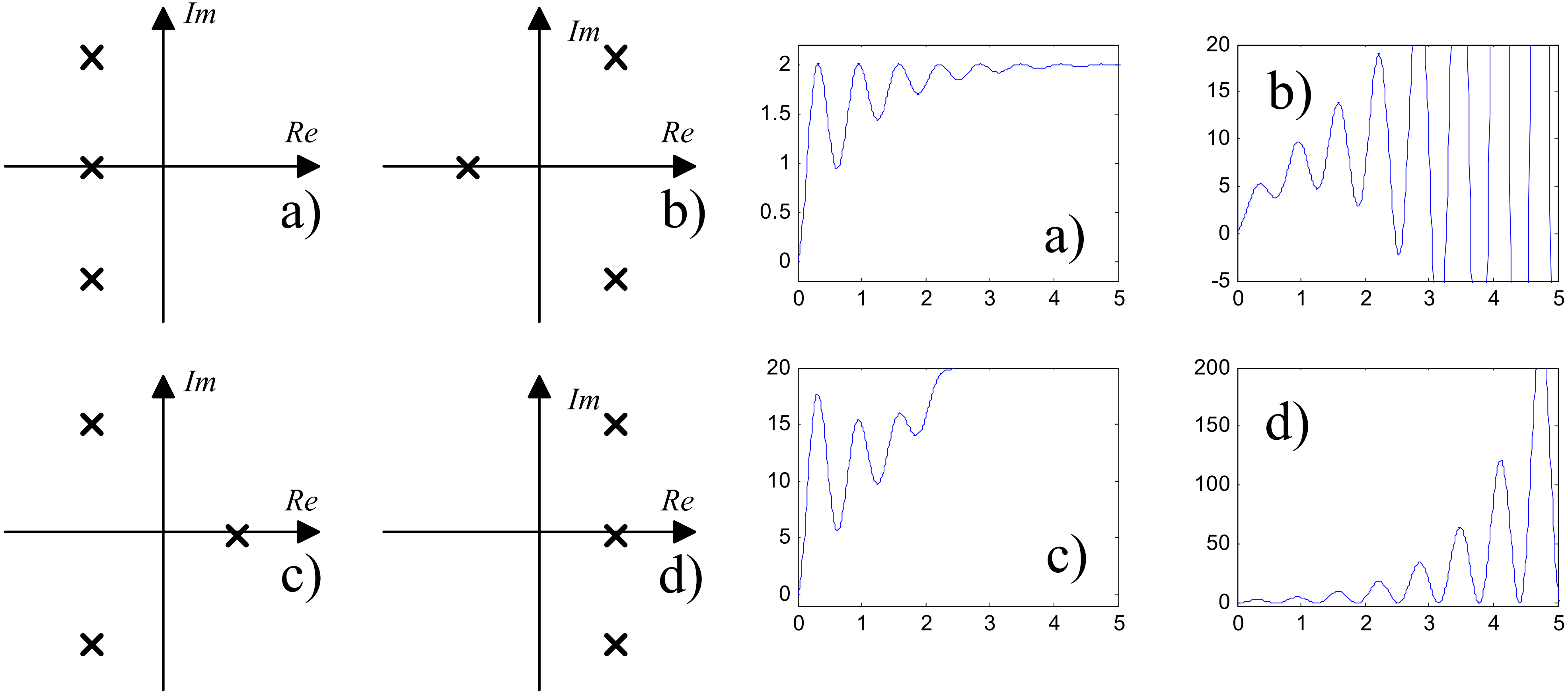
$$G(s) = \frac{K\omega_n^2}{(\tau s + 1)(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

  - The behavior of this system is a weighted sum of second-order behavior and first-order behavior:

$$G(s) = \frac{K\omega_n^2}{(\tau s + 1)(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{\tau s + 1} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

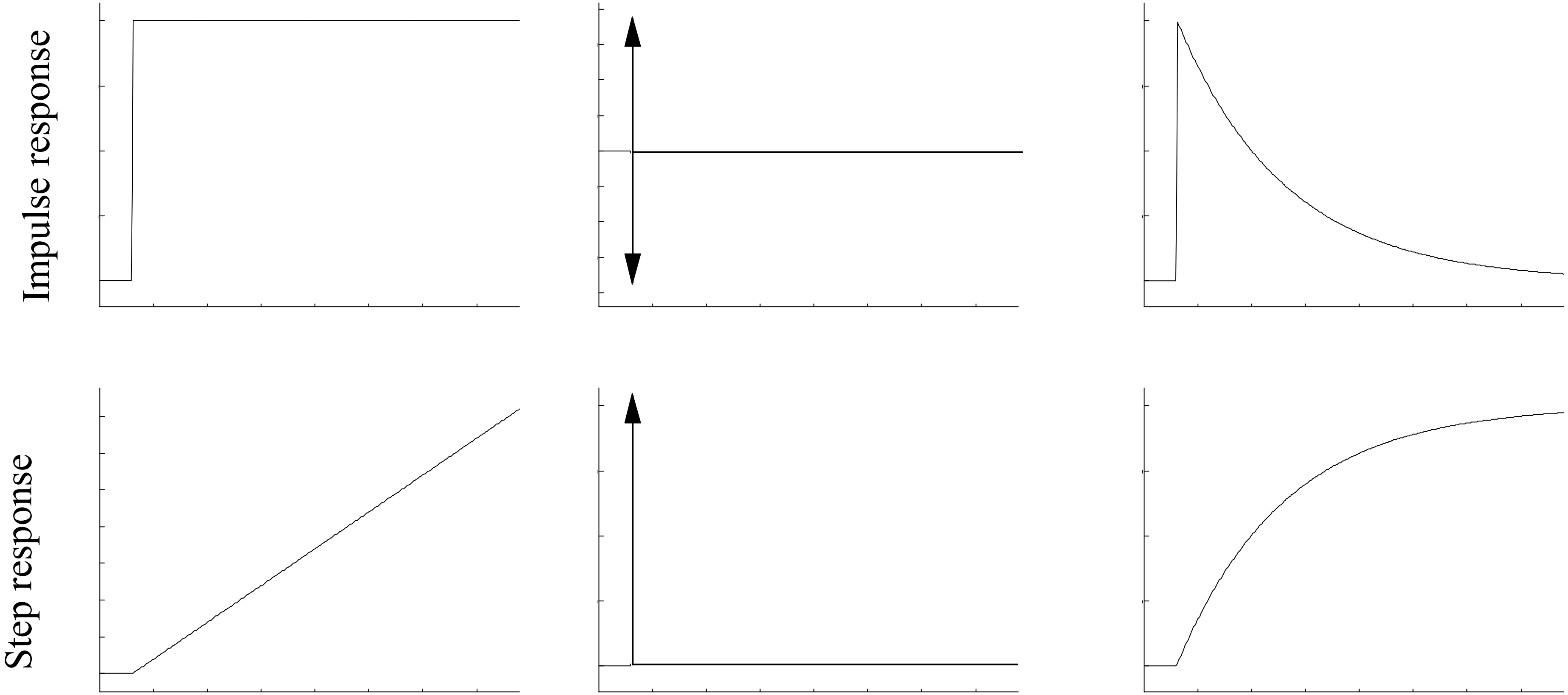
Higher-order Models

- Below are system pole zero patterns and step responses with certain parameter values



Simple system responses

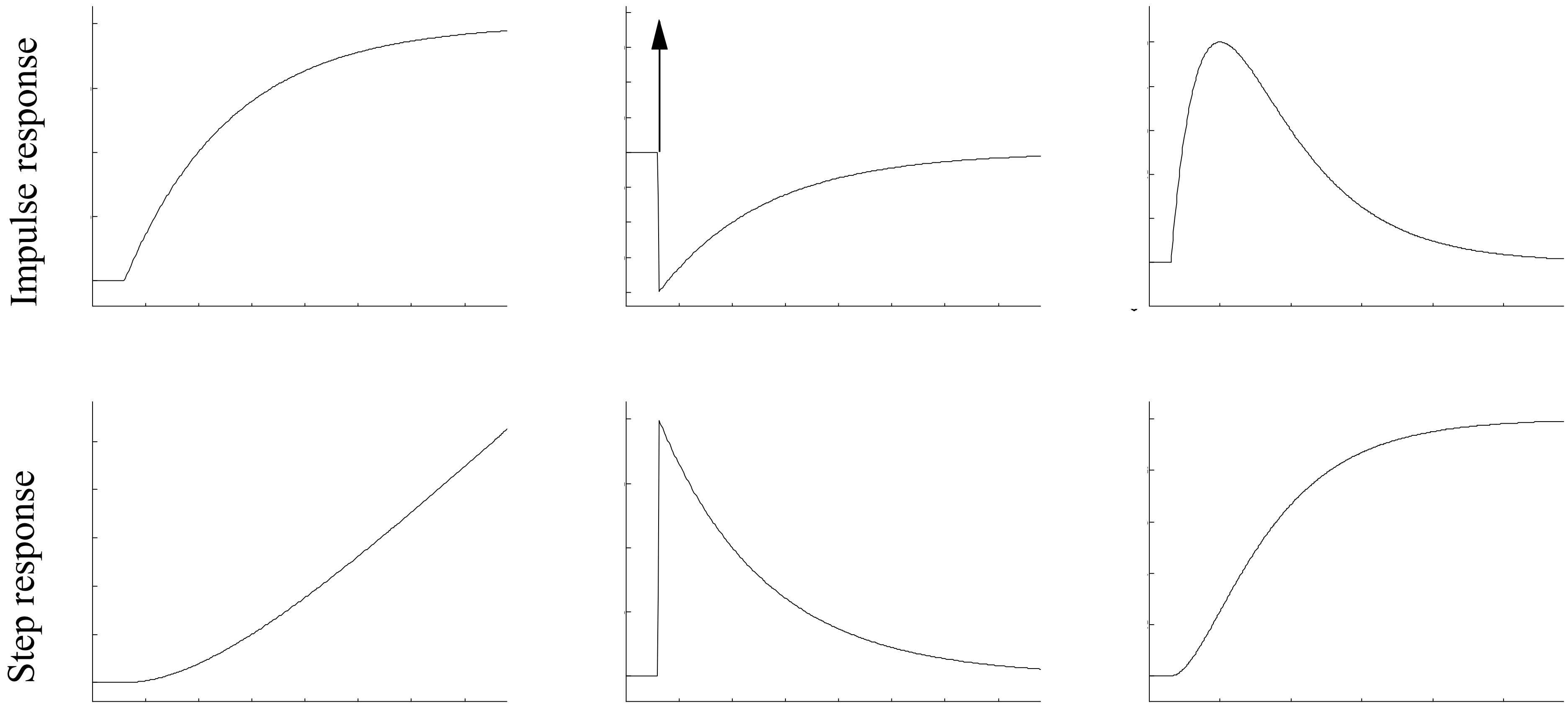
$$\frac{K}{s} \qquad\qquad Ks \qquad\qquad \frac{K}{\tau s+1}$$



$$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$$

Simple system responses

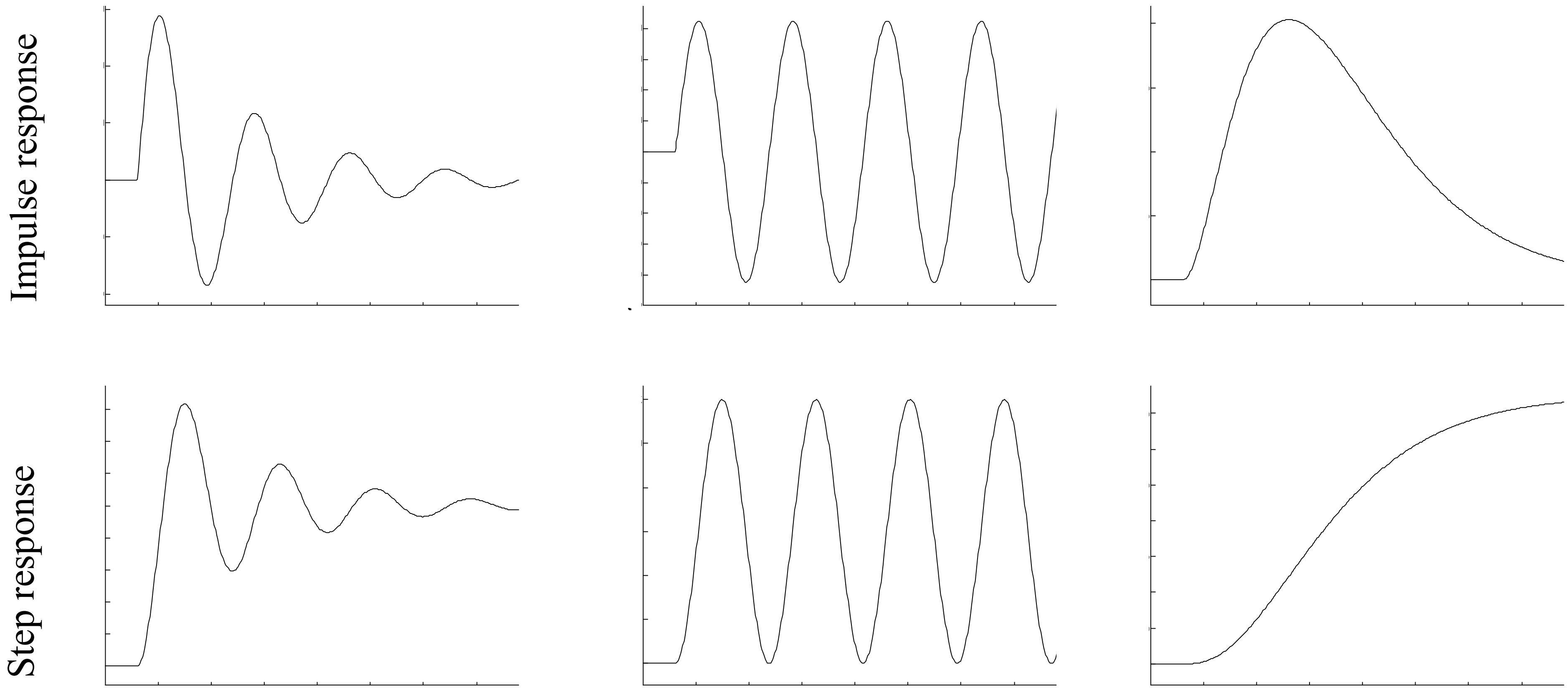
$$\frac{K}{s(\tau s+1)} \qquad \frac{Ks}{\tau s+1} \qquad \frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$



$$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$$

Simple system responses

$$\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \frac{K\omega_n^2}{s^2 + \omega_n^2} \qquad \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)}$$



$$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$$

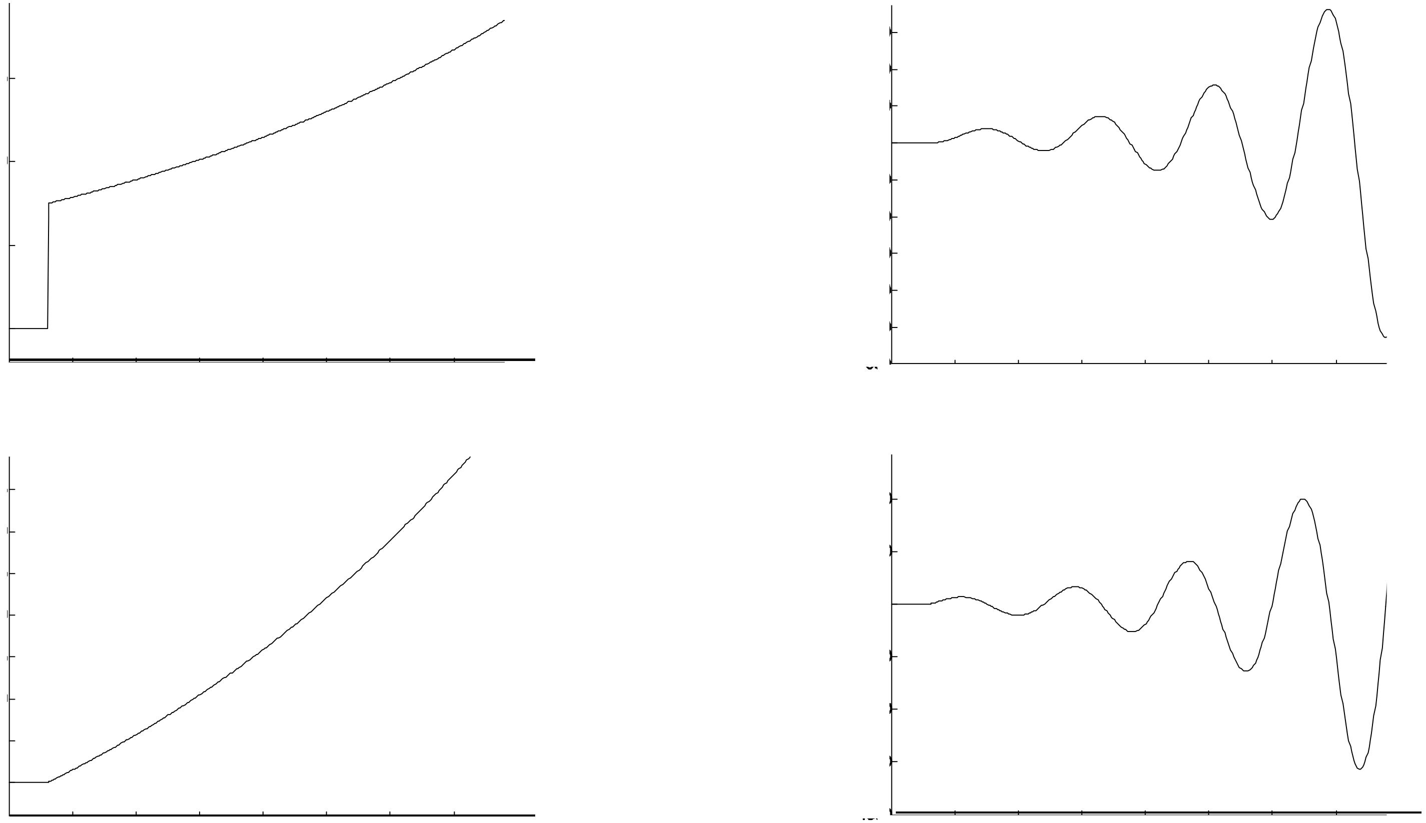
Simple system responses

$$\frac{K}{\tau s - 1}$$

$$\frac{K\omega_n^2}{s^2 - 2\zeta\omega_n s + \omega_n^2}$$

Impulse response

Step response



$$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$$