Lecture 5: Unscented Kalman filter, Gaussian Filter, GHKF and CKF

Simo Särkkä

February 5, 2020

Learning Outcomes

1. Summary of the Last Lecture
2. Unscented Transform
3. Unscented Kalman Filter (UKF)
4. Gaussian Filter
5. Gauss-Hermite Kalman Filter (GHKF)
6. Cubature Kalman Filter (CKF)
7. Summary and Demonstration

Summary of the Last Lecture

- EKF and SLF can be applied to filtering models of the form

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \\
\mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k,
\end{aligned}
$$

- EKF is based on Taylor series expansions of **f** and **h**.
  - Advantages: Simple, intuitive, computationally efficient
  - Disadvantages: Local approximation, differentiability requirements, only for Gaussian noises.
- SLF is based on statistical linearization:
  - Advantages: Global approximation, no differentiability requirements, computationally efficient
  - Disadvantages: Closed form computation of expectations, only for Gaussian noises.
  - But, there is a connection to sigma-point filters (e.g., UKF).

Linearization Based Gaussian Approximation

- Problem: Determine the mean and covariance of y:

$$x \sim \mathrm{N}(\mu, \sigma^2)$$
$$y = \sin(x)$$

- Linearization based approximation:

$$y = \sin(\mu) + \frac{\partial \sin(\mu)}{\partial \mu}(x - \mu) + \dots$$

which gives

$$\begin{aligned} \mathrm{E}[y] &\approx \mathrm{E}[\sin(\mu) + \cos(\mu)(x - \mu)] = \sin(\mu) \\ \mathrm{Cov}[y] &\approx \mathrm{E}[(\sin(\mu) + \cos(\mu)(x - \mu) - \sin(\mu))^2] = \cos^2(\mu)\, \sigma^2. \end{aligned}$$

- Form 3 sigma points as follows:

$$\begin{aligned} \mathcal{X}^{(0)} &= \mu \\ \mathcal{X}^{(1)} &= \mu + \sigma \\ \mathcal{X}^{(2)} &= \mu - \sigma. \end{aligned}$$

- Let's select some weights $W^{(0)}, W^{(1)}, W^{(2)}$ such that the original mean and variance can be recovered by

$$\begin{aligned} \mu &= \sum_i W^{(i)} \, \mathcal{X}^{(i)} \\ \sigma^2 &= \sum_i W^{(i)} \, (\mathcal{X}^{(i)} - \mu)^2. \end{aligned}$$

Principle of Unscented Transform [2/3]

- We use the same formula for approximating the moments of $y = \sin(x)$ as follows:

$$\mu_y = \sum_i W^{(i)} \sin(\mathcal{X}^{(i)})$$
$$\sigma_y^2 = \sum_i W^{(i)} (\sin(\mathcal{X}^{(i)}) - \mu)^2.$$

- For vectors $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ the generalization of standard deviation σ is the Cholesky factor $\mathbf{L} = \sqrt{\mathbf{P}}$:

$$\mathbf{P} = \mathbf{L}\,\mathbf{L}^\mathsf{T}.$$

- The sigma points can be formed using columns of $\mathbf{L}$ (here c is a suitable positive constant):

$$\mathcal{X}^{(0)} = \mathbf{m}$$
$$\mathcal{X}^{(i)} = \mathbf{m} + c\,\mathbf{L}_i$$
$$\mathcal{X}^{(n+i)} = \mathbf{m} - c\,\mathbf{L}_i$$

- For transformation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ the approximation is:

$$\boldsymbol{\mu}_y = \sum_i W^{(i)} \, \mathbf{g}(\mathcal{X}^{(i)})$$
$$\Sigma_y = \sum_i W^{(i)} \, (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y) \, (\mathbf{g}(\mathcal{X}^{(i)}) - \boldsymbol{\mu}_y)^\mathsf{T}.$$

- It is convenient to define transformed sigma points:

$$\mathcal{Y}^{(i)} = \mathbf{g}(\mathcal{X}^{(i)})$$

- Joint moments of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ are then approximated as

$$\mathrm{E}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix}\right] \approx \sum_i W^{(i)} \begin{pmatrix} \mathcal{X}^{(i)} \\ \mathcal{Y}^{(i)} \end{pmatrix} = \begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_y \end{pmatrix}$$

$$\mathrm{Cov}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix}\right]$$
$$\approx \sum_i W^{(i)} \begin{pmatrix} (\mathcal{X}^{(i)} - \mathbf{m}) \, (\mathcal{X}^{(i)} - \mathbf{m})^\mathsf{T} & (\mathcal{X}^{(i)} - \mathbf{m}) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y)^\mathsf{T} \\ (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y) \, (\mathcal{X}^{(i)} - \mathbf{m})^\mathsf{T} & (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_y)^\mathsf{T} + \mathbf{Q} \end{pmatrix}$$

Unscented transform

The unscented transform approximation to the joint distribution of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_U \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_U \\ \mathbf{C}_U^\mathsf{T} & \mathbf{S}_U \end{pmatrix} \right),$$

where the sub-matrices are formed as follows:

1. Form the sigma points as

$$\begin{aligned} \mathcal{X}^{(0)} &= \mathbf{m} \\ \mathcal{X}^{(i)} &= \mathbf{m} + \sqrt{n+\lambda} \left[\sqrt{\mathbf{P}} \right]_i \\ \mathcal{X}^{(i+n)} &= \mathbf{m} - \sqrt{n+\lambda} \left[\sqrt{\mathbf{P}} \right]_i, \quad i = 1, \ldots, n \end{aligned}$$

Unscented Transform [2/3]

Unscented transform (cont.)

2. Propagate the sigma points through $\mathbf{g}(\cdot)$:

$$\mathcal{Y}^{(i)} = \mathbf{g}(\mathcal{X}^{(i)}), \quad i = 0, \ldots, 2n.$$

3. The sub-matrices are then given as:

$$\begin{aligned}
\boldsymbol{\mu}_U &= \sum_{i=0}^{2n} W_i^{(m)} \, \mathcal{Y}^{(i)} \\
\mathbf{S}_U &= \sum_{i=0}^{2n} W_i^{(c)} \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U)^\mathsf{T} + \mathbf{Q} \\
\mathbf{C}_U &= \sum_{i=0}^{2n} W_i^{(c)} \, (\mathcal{X}^{(i)} - \mathbf{m}) \, (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_U)^\mathsf{T}.
\end{aligned}$$

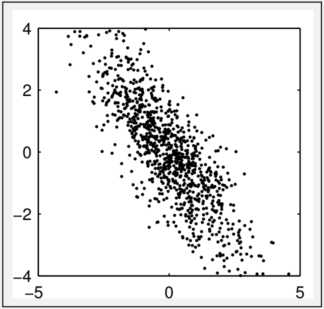
Unscented transform (cont.)

- λ is a scaling parameter defined as $\lambda = \alpha^2\,(n + \kappa) - n$.
- α and κ determine the spread of the sigma points.
- Weights $W_i^{(m)}$ and $W_i^{(c)}$ are given as follows:

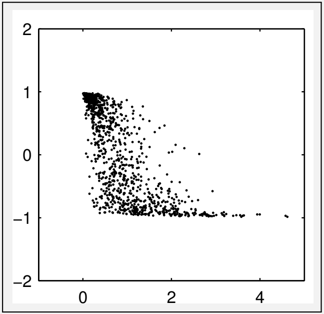
$$
\begin{aligned}
W_0^{(m)} &= \lambda/(n+\lambda) \\
W_0^{(c)} &= \lambda/(n+\lambda) + (1 - \alpha^2 + \beta) \\
W_i^{(m)} &= 1/\{2(n+\lambda)\}, \quad i = 1, \ldots, 2n \\
W_i^{(c)} &= 1/\{2(n+\lambda)\}, \quad i = 1, \ldots, 2n,
\end{aligned}
$$

- β can be used for incorporating prior information on the (non-Gaussian) distribution of $\mathbf{x}$.

Linearization/UT Example



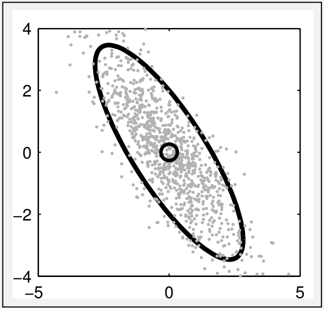
$\Longrightarrow$



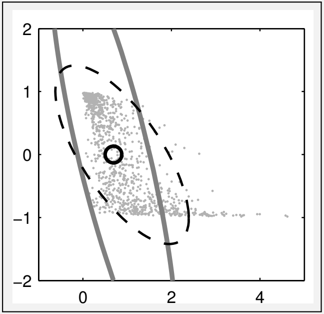
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \right)$$

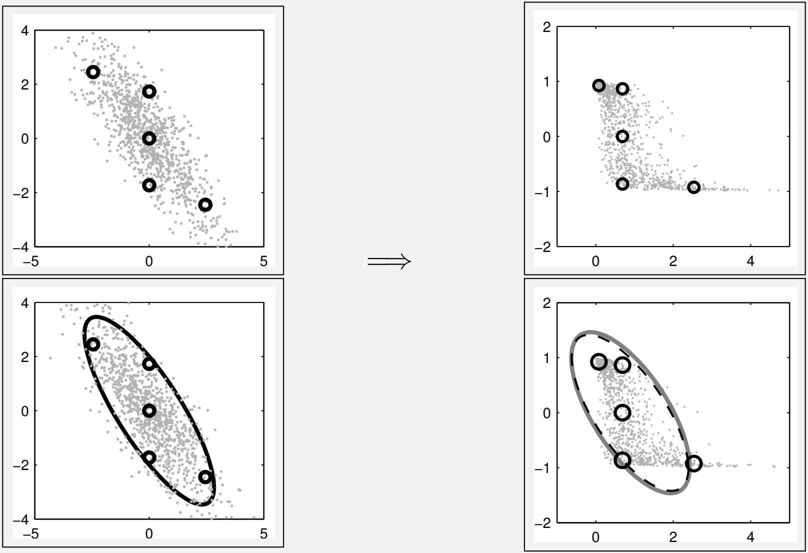
$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \exp(-y_1), & y_1(0) &= x_1 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\frac{1}{2} y_2^3, & y_2(0) &= x_2 \end{aligned}$$

Linearization Approximation



$\Longrightarrow$




4
2
0
−2
−4
−5
0
5
⟹
2
1
0
−1
−2
0
2
4

Unscented Kalman Filter (UKF): Derivation [1/4]

- Assume that the filtering distribution of previous step is Gaussian

$$p(\mathbf{x}_{k-1} \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}(\mathbf{x}_{k-1} \mid \mathbf{m}_{k-1}, \mathbf{P}_{k-1})$$

- The joint distribution of $\mathbf{x}_{k-1}$ and $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}$ can be approximated with UT as Gaussian

$$p(\mathbf{x}_{k-1}, \mathbf{x}_k \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_{k-1} \\ \mathbf{x}_k \end{bmatrix} \,\middle|\, \begin{pmatrix} \mathbf{m}'_1 \\ \mathbf{m}'_2 \end{pmatrix}, \begin{pmatrix} \mathbf{P}'_{11} & \mathbf{P}'_{12} \\ (\mathbf{P}'_{12})^\mathsf{T} & \mathbf{P}'_{22} \end{pmatrix} \right),$$

  as follows.
- Form the sigma points $\mathcal{X}^{(i)}$ of $\mathbf{x}_{k-1} \sim \mathrm{N}(\mathbf{m}_{k-1}, \mathbf{P}_{k-1})$ and compute the transformed sigma points as $\hat{\mathcal{X}}^{(i)} = \mathbf{f}(\mathcal{X}^{(i)})$.
- The expected values can now be expressed as:

$$\mathbf{m}'_1 = \mathbf{m}_{k-1}$$
$$\mathbf{m}'_2 = \sum_i W_i^{(m)} \, \hat{\mathcal{X}}^{(i)}$$

- The blocks of covariance can be expressed as:

$$
\begin{aligned}
\mathbf{P}'_{11} &= \mathbf{P}_{k-1} \\
\mathbf{P}'_{12} &= \sum_i W_i^{(c)} (\mathcal{X}^{(i)} - \mathbf{m}_{k-1}) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2)^\mathsf{T} \\
\mathbf{P}'_{22} &= \sum_i W_i^{(c)} (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}'_2)^\mathsf{T} + \mathbf{Q}_{k-1}
\end{aligned}
$$

- The prediction mean and covariance of $\mathbf{x}_k$ are then $\mathbf{m}'_2$ and $\mathbf{P}'_{22}$, and thus we get

$$
\begin{aligned}
\mathbf{m}_k^- &= \sum_i W_i^{(m)} \, \hat{\mathcal{X}}^{(i)} \\
\mathbf{P}_k^- &= \sum_i W_i^{(c)} (\hat{\mathcal{X}}^{(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{X}}^{(i)} - \mathbf{m}_k^-)^\mathsf{T} + \mathbf{Q}_{k-1}
\end{aligned}
$$

Unscented Kalman Filter (UKF): Derivation [3/4]

- For the joint distribution of $\mathbf{x}_k$ and $\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k$ we similarly get

$$p(\mathbf{x}_k, \mathbf{y}_k, \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} \;\middle|\; \begin{pmatrix} \mathbf{m}_1'' \\ \mathbf{m}_2'' \end{pmatrix}, \begin{pmatrix} \mathbf{P}_{11}'' & \mathbf{P}_{12}'' \\ (\mathbf{P}_{12}'')^\mathsf{T} & \mathbf{P}_{22}'' \end{pmatrix} \right),$$

- If $\mathcal{X}^{-(i)}$ are the sigma points of $\mathbf{x}_k \sim \mathrm{N}(\mathbf{m}_k^-, \mathbf{P}_k^-)$ and $\hat{\mathcal{Y}}^{(i)} = \mathbf{h}(\mathcal{X}^{-(i)})$, we get:

$$\begin{aligned}
\mathbf{m}_1'' &= \mathbf{m}_k^- \\
\mathbf{m}_2'' &= \sum_i W_i^{(m)} \, \hat{\mathcal{Y}}^{(i)} \\
\mathbf{P}_{11}'' &= \mathbf{P}_k^- \\
\mathbf{P}_{12}'' &= \sum_i W_i^{(c)} (\mathcal{X}^{-(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{Y}}^{(i)} - \mathbf{m}_2'')^\mathsf{T} \\
\mathbf{P}_{22}'' &= \sum_i W_i^{(c)} (\hat{\mathcal{Y}}^{(i)} - \mathbf{m}_2'') \, (\hat{\mathcal{Y}}^{(i)} - \mathbf{m}_2'')^\mathsf{T} + \mathbf{R}_k
\end{aligned}$$

Unscented Kalman Filter (UKF): Derivation [4/4]

- Recall that if

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\mathsf{T} & \mathbf{B} \end{pmatrix} \right),$$

then

$$\mathbf{x} \mid \mathbf{y} \sim \mathrm{N}(\mathbf{a} + \mathbf{C}\,\mathbf{B}^{-1}\,(\mathbf{y} - \mathbf{b}), \mathbf{A} - \mathbf{C}\,\mathbf{B}^{-1}\mathbf{C}^\mathsf{T}).$$

- Thus we get the conditional mean and covariance:

$$\begin{aligned} \mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{P}_{12}''\,(\mathbf{P}_{22}'')^{-1}(\mathbf{y}_k - \mathbf{m}_2'') \\ \mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{P}_{12}''\,(\mathbf{P}_{22}'')^{-1}\,(\mathbf{P}_{12}'')^\mathsf{T}. \end{aligned}$$

Unscented Kalman Filter (UKF): Algorithm [1/4]

Unscented Kalman filter: Prediction step

1. Form the sigma points:

$$
\begin{aligned}
\mathcal{X}_{k-1}^{(0)} &= \mathbf{m}_{k-1}, \\
\mathcal{X}_{k-1}^{(i)} &= \mathbf{m}_{k-1} + \sqrt{n+\lambda}\,\left[\sqrt{\mathbf{P}_{k-1}}\right]_i \\
\mathcal{X}_{k-1}^{(i+n)} &= \mathbf{m}_{k-1} - \sqrt{n+\lambda}\,\left[\sqrt{\mathbf{P}_{k-1}}\right]_i, \quad i = 1, \ldots, n.
\end{aligned}
$$

2. Propagate the sigma points through the dynamic model:

$$
\hat{\mathcal{X}}_k^{(i)} = \mathbf{f}(\mathcal{X}_{k-1}^{(i)}). \quad i = 0, \ldots, 2n.
$$

Unscented Kalman Filter (UKF): Algorithm [2/4]

Unscented Kalman filter: Prediction step (cont.)

3. Compute the predicted mean and covariance:

$$\mathbf{m}_k^- = \sum_{i=0}^{2n} W_i^{(m)} \, \hat{\mathcal{X}}_k^{(i)}$$

$$\mathbf{P}_k^- = \sum_{i=0}^{2n} W_i^{(c)} \, (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)^\mathsf{T} + \mathbf{Q}_{k-1}.$$

Unscented Kalman Filter (UKF): Algorithm [3/4]

Unscented Kalman filter: Update step

1. Form the sigma points:

$$
\begin{aligned}
\mathcal{X}_k^{-(0)} &= \mathbf{m}_k^-, \\
\mathcal{X}_k^{-(i)} &= \mathbf{m}_k^- + \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_k^-}\right]_i \\
\mathcal{X}_k^{-(i+n)} &= \mathbf{m}_k^- - \sqrt{n+\lambda}\left[\sqrt{\mathbf{P}_k^-}\right]_i, \quad i = 1,\ldots,n.
\end{aligned}
$$

2. Propagate sigma points through the measurement model:

$$
\hat{\mathcal{Y}}_k^{(i)} = \mathbf{h}(\mathcal{X}_k^{-(i)}), \quad i = 0,\ldots,2n.
$$

Unscented Kalman Filter (UKF): Algorithm [4/4]

Unscented Kalman filter: Update step (cont.)

3. Compute the following:

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \sum_{i=0}^{2n} W_i^{(m)} \, \hat{\mathcal{Y}}_k^{(i)} \\
\mathbf{S}_k &= \sum_{i=0}^{2n} W_i^{(c)} \, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k) \, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^\mathsf{T} + \mathbf{R}_k \\
\mathbf{C}_k &= \sum_{i=0}^{2n} W_i^{(c)} \, (\mathcal{X}_k^{-(i)} - \mathbf{m}_k^-) \, (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^\mathsf{T} \\
\mathbf{K}_k &= \mathbf{C}_k \, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k \, [\mathbf{y}_k - \boldsymbol{\mu}_k] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k \, \mathbf{S}_k \, \mathbf{K}_k^\mathsf{T}.
\end{aligned}
$$

Unscented Kalman Filter (UKF): Advantages

- No closed form derivatives or expectations needed.
- Not a local approximation, but based on values on a larger area.
- Functions $\mathbf{f}$ and $\mathbf{h}$ do not need to be differentiable.
- Theoretically, captures higher order moments of distribution than linearization — the mean is correct for up to third order monomials.

Unscented Kalman Filter (UKF): Disadvantage

- Not a truly global approximation, based on a small set of trial points.
- Does not work well with nearly singular covariances, i.e., with nearly deterministic systems.
- Requires more computations than EKF or SLF, e.g., Cholesky factorizations on every step.
- The covariance computation is exact only for linear functions.
- Can only be applied to models driven by Gaussian noises.

Gaussian Moment Matching [1/2]

- Consider the transformation of $\mathbf{x}$ into $\mathbf{y}$:

$$\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}).$$

- Form Gaussian approximation to $(\mathbf{x}, \mathbf{y})$ by directly approximating the integrals:

$$\mu_M = \int \mathbf{g}(\mathbf{x}) \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{S}_M = \int (\mathbf{g}(\mathbf{x}) - \mu_M) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^\mathsf{T} \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{C}_M = \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^\mathsf{T} \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}.$$

Gaussian moment matching

The moment matching based Gaussian approximation to the joint distribution of $\mathbf{x}$ and the transformed random variable $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is given as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_M \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_M \\ \mathbf{C}_M^\mathsf{T} & \mathbf{S}_M \end{pmatrix} \right),$$

where

$$\begin{aligned}
\boldsymbol{\mu}_M &= \int \mathbf{g}(\mathbf{x}) \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\
\mathbf{S}_M &= \int (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^\mathsf{T} \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} + \mathbf{Q} \\
\mathbf{C}_M &= \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^\mathsf{T} \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}.
\end{aligned}$$

Connection with Statistical Linearization

- An alternative way of writing the moment matching:

$$
\begin{aligned}
\boldsymbol{\mu}_M &= \mathrm{E}[\mathbf{g}(\mathbf{x})] \\
\mathbf{S}_M &= \mathrm{Cov}[\mathbf{g}(\mathbf{x})] \\
\mathbf{C}_M &= \mathrm{E}[\mathbf{g}(\mathbf{x})\,\delta\mathbf{x}^\mathsf{T}]^\mathsf{T}.
\end{aligned}
$$

  where $\delta\mathbf{x} = \mathbf{x} - \mathbf{m}$.

- Differs from statistical linearization only in the **S**-terms:

$$
\begin{aligned}
\boldsymbol{\mu}_S &= \mathrm{E}[\mathbf{g}(\mathbf{x})] \\
\mathbf{S}_S &= \mathrm{E}[\mathbf{g}(\mathbf{x})\,\delta\mathbf{x}^\mathsf{T}]\,\mathbf{P}^{-1}\;\mathrm{E}[\mathbf{g}(\mathbf{x})\,\delta\mathbf{x}^\mathsf{T}]^\mathsf{T} + \mathbf{Q} \\
\mathbf{C}_S &= \mathrm{E}[\mathbf{g}(\mathbf{x})\,\delta\mathbf{x}^\mathsf{T}]^\mathsf{T}.
\end{aligned}
$$

- The Gaussian moment matching is equivalent to so called statistical linear regression.

Gaussian Filter [1/3]

Gaussian filter prediction

Compute the following Gaussian integrals:

$$
\begin{aligned}
\mathbf{m}_k^- &= \int \mathbf{f}(\mathbf{x}_{k-1})\, \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1})\, d\mathbf{x}_{k-1} \\
\mathbf{P}_k^- &= \int (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)\,(\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)^\mathsf{T} \\
&\qquad \times \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1})\, d\mathbf{x}_{k-1} + \mathbf{Q}_{k-1}.
\end{aligned}
$$

Gaussian Filter [2/3]

Gaussian filter update

1. Compute the following Gaussian integrals:

$$\begin{aligned}
\boldsymbol{\mu}_k &= \int \mathbf{h}(\mathbf{x}_k)\; \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k \\
\mathbf{S}_k &= \int (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)\,(\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^\mathsf{T}\; \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k + \mathbf{R}_k \\
\mathbf{C}_k &= \int (\mathbf{x}_k - \mathbf{m}_k^-)\,(\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^\mathsf{T}\; \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k.
\end{aligned}$$

2. Then compute the following:

$$\begin{aligned}
\mathbf{K}_k &= \mathbf{C}_k\, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\,(\mathbf{y}_k - \boldsymbol{\mu}_k) \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^\mathsf{T}.
\end{aligned}$$

Gaussian Filter [3/3]

- Special case of assumed density filtering (ADF).
- Multidimensional Gauss-Hermite quadrature ⇒ Gauss Hermite Kalman filter (GHKF).
- Cubature integration ⇒ Cubature Kalman filter (CKF).
- Monte Carlo integration ⇒ Monte Carlo Kalman filter (MCKF).
- Gaussian process / Bayes-Hermite Kalman filter: Form Gaussian process regression model from set of sample points and integrate the approximation.
- Linearization, unscented transform, central differences, divided differences can be considered as special cases.

- One-dimensional Gauss-Hermite quadrature of order p:

$$\int_{-\infty}^{\infty} g(x)\, \mathrm{N}(x \mid 0, 1)\, dx \approx \sum_{i=1}^{p} W^{(i)} g(x^{(i)}),$$

- $\xi^{(i)}$ are roots of pth order Hermite polynomial:

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \ldots \end{aligned}$$

- The weights are $W^{(i)} = p!/(p^2\, [H_{p-1}(\xi^{(i)})]^2)$.
- Exact for polynomials up to order $2p - 1$.

- Multidimensional integrals can be approximated as:

$$
\begin{aligned}
&\int \mathbf{g}(\mathbf{x}) \; \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\
&= \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \; \mathrm{N}(\xi \,|\, \mathbf{0}, \mathbf{I}) \, d\xi \\
&= \int \cdots \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \; \mathrm{N}(\xi_1 \,|\, 0, 1) \, d\xi_1 \times \cdots \times \mathrm{N}(\xi_n \,|\, 0, 1) \, d\xi_n \\
&\approx \sum_{i_1, \ldots, i_n} W^{(i_1)} \times \cdots \times W^{(i_n)} \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi^{(i_1, \ldots, i_n)}).
\end{aligned}
$$

- Needs p^n evaluation points.
- Gauss-Hermite Kalman filter (GHKF) uses this for evaluation of the Gaussian integrals.

- Postulate symmetric integration rule:

$$\int \mathbf{g}(\xi)\; \mathrm{N}(\xi \,|\, \mathbf{0}, \mathbf{I})\, d\xi \approx W \sum_i \mathbf{g}(c\,\mathbf{u}^{(i)}),$$

- The points $\mathbf{u}^{(i)}$ belong to the symmetric set $[\mathbf{1}]$ with generator $(1, 0, \ldots, 0)$:

$$[\mathbf{1}] = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \right\}$$

- W is a weight and c is a parameter yet to be determined.

Spherical Cubature Integration [2/3]

- Due to symmetry, all odd orders are integrated exactly.
- We only need to match the following moments:

$$\int \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi = 1$$

$$\int \xi_j^2 \, \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi = 1$$

- Thus we get the equations

$$W \sum_i 1 = W \, 2n = 1$$

$$W \sum_i [c \, u_j^{(i)}]^2 = W \, 2c^2 = 1$$

- Hence the following rule is exact up to third degree:

$$\int \mathbf{g}(\xi) \; \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi \approx \frac{1}{2n} \sum_i \mathbf{g}(\sqrt{n} \, \mathbf{u}^{(i)}).$$

Spherical Cubature Integration [3/3]

- The resulting Gaussian integral rule:

$$\begin{aligned}
&\int \mathbf{g}(\mathbf{x}) \; \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\
&= \int \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi) \; \mathrm{N}(\xi \mid \mathbf{0}, \mathbf{I}) \, d\xi \\
&\approx \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{g}(\mathbf{m} + \sqrt{\mathbf{P}} \, \xi^{(i)}),
\end{aligned}$$

where

$$\xi^{(i)} = \begin{cases} \sqrt{n} \, \mathbf{e}_i & , \quad i = 1, \ldots, n \\ -\sqrt{n} \, \mathbf{e}_{i-n} & , \quad i = n+1, \ldots, 2n, \end{cases}$$

where $\mathbf{e}_i$ denotes a unit vector to the direction of coordinate axis i.

Cubature Kalman Filter (CKF) [1/4]

Cubature Kalman filter: Prediction step

1. Form the sigma points as:

$$\mathcal{X}_{k-1}^{(i)} = \mathbf{m}_{k-1} + \sqrt{\mathbf{P}_{k-1}}\,\xi^{(i)} \qquad i = 1, \ldots, 2n.$$

2. Propagate the sigma points through the dynamic model:

$$\hat{\mathcal{X}}_k^{(i)} = \mathbf{f}(\mathcal{X}_{k-1}^{(i)}). \quad i = 1 \ldots 2n.$$

3. Compute the predicted mean and covariance:

$$\mathbf{m}_k^- = \frac{1}{2n} \sum_{i=1}^{2n} \hat{\mathcal{X}}_k^{(i)}$$

$$\mathbf{P}_k^- = \frac{1}{2n} \sum_{i=1}^{2n} (\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)\,(\hat{\mathcal{X}}_k^{(i)} - \mathbf{m}_k^-)^\mathsf{T} + \mathbf{Q}_{k-1}.$$

Cubature Kalman Filter (CKF) [2/4]

Cubature Kalman filter: Update step

1. Form the sigma points:

$$\mathcal{X}_k^{-(i)} = \mathbf{m}_k^- + \sqrt{\mathbf{P}_k^-}\,\xi^{(i)}, \qquad i = 1, \ldots, 2n.$$

2. Propagate sigma points through the measurement model:

$$\hat{\mathcal{Y}}_k^{(i)} = \mathbf{h}(\mathcal{X}_k^{-(i)}), \quad i = 1 \ldots 2n.$$

Cubature Kalman Filter (CKF) [3/4]

Cubature Kalman filter: Update step (cont.)

3. Compute the following:

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \frac{1}{2n} \sum_{i=1}^{2n} \hat{\mathcal{Y}}_k^{(i)} \\
\mathbf{S}_k &= \frac{1}{2n} \sum_{i=1}^{2n} (\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)\,(\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^\mathsf{T} + \mathbf{R}_k \\
\mathbf{C}_k &= \frac{1}{2n} \sum_{i=1}^{2n} (\mathcal{X}_k^{-(i)} - \mathbf{m}_k^-)\,(\hat{\mathcal{Y}}_k^{(i)} - \boldsymbol{\mu}_k)^\mathsf{T} \\
\mathbf{K}_k &= \mathbf{C}_k\,\mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\,[\mathbf{y}_k - \boldsymbol{\mu}_k] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\,\mathbf{S}_k\,\mathbf{K}_k^\mathsf{T}.
\end{aligned}
$$

Cubature Kalman Filter (CKF) [4/4]

- Cubature Kalman filter (CKF) is a special case of UKF with $\alpha = 1$, $\beta = 0$, and $\kappa = 0$ – the mean weight becomes zero with these choices.
- Rule is exact for third order polynomials (multinomials) – note that third order Gauss-Hermite is exact for fifth order polynomials.
- UKF was also originally derived using similar way, but is a bit more general.
- Very easy algorithm to implement – you can even recall the rule by heart.

Summary

- Unscented transform (UT) approximates transformations of Gaussian variables by propagating sigma points through the non-linearity.
- In UT the mean and covariance are approximated as linear combination of the sigma points.
- The unscented Kalman filter uses unscented transform for computing the approximate means and covariance in non-linear filtering problems.
- A non-linear transformation can also be approximated with Gaussian moment matching.
- Gaussian filter is based on matching the moments with numerical integration $\Rightarrow$ many kinds of Kalman filters.
- Gauss-Hermite Kalman filter (GHKF) and Cubature Kalman filter (CKF) are examples of them.

Unscented/Cubature Kalman Filter (UKF/CKF): Example

- Recall the discretized pendulum model

$$\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k-1}^1 + x_{k-1}^2 \, \Delta t \\ x_{k-1}^2 - g \, \sin(x_{k-1}^1) \, \Delta t \end{pmatrix}}_{\mathbf{f}(\mathbf{x_{k-1}})} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}$$

$$y_k = \underbrace{\sin(x_k^1)}_{\mathbf{h}(\mathbf{x_k})} + r_k,$$

- $\Rightarrow$ *Matlab demonstration*