Filter structures
Basic concepts

- Linear-phase FIR filters, types I-IV
  - Odd/even length, symmetrical/anti-symmetrical

\[
H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots + h_{N-1} z^{-(N-1)}
\]

\[
h_k = \pm h_{N-1-k}, \quad k = 0, 1, \ldots, N - 1
\]

\[
H(z) = \pm z^{-(N-1)} H(z^{-1})
\]

- Group delay

\[
\tau_H(\omega) = \frac{-d\theta(\omega)}{d\omega}, \quad \theta(\omega) = \arg\{H(e^{j\omega})\}
\]

- Magnitude response $|H(e^{j\omega})|$
Basic concepts

• Translation of filter’s pass-band
  • Modulation of filter coefficients

\[ H_k(z) = H_0(z e^{-j2\pi k/M}), \quad 0 \leq k \leq M - 1 \]
\[ H_k(e^{j\omega}) = H_0(e^{j(\omega-2\pi k/M)}), \quad 0 \leq k \leq M - 1 \]
\[ h_k[n] = h_0[n]e^{j2\pi nk/M}, \quad 0 \leq k \leq M - 1 \]

• Complementary filter \( G(z) = z^{-N/2} - H(z) \)
Basic concepts

- **Zero-phase/amplitude response**
  - Impossible to implement in practice, but easy to implement when causality is relaxed
  - Unlike magnitude response, can be negative

- **Zero-phase response of symmetric even-order N (Type I) FIR filter**
  \[
  H(e^{j\omega}) = e^{-j\omega N/2} \tilde{H}(\omega)
  \]
  \[
  \tilde{H}(\omega) = h\left[\frac{N}{2}\right] + 2 \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \cos(n\omega)
  \]

- **Zero-phase response of Type I-IV FIR filters**
  \[
  H(e^{j\omega}) = e^{-j\omega N/2} e^{j\beta} \tilde{H}(\omega)
  \]
Polyphase decomposition
Polyphase decomposition

- Consider the z-transform of sequence $x[n]$
- $X(z)$ can be rewritten as

$$X(z) = \sum_{k=0}^{M-1} z^{-k} X_k(z^M)$$

$$X_k(z) = \sum_{n=-\infty}^{\infty} x_k[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[Mn+k]z^{-n}, \ 0 \leq k \leq M - 1$$

- Subsequences $x_k[n]$ are called polyphase components of $x[n]$
- Functions $X_k(z)$ are called polyphase components of $X(z)$
Type 1 polyphase decomposition

- Polyphase decomposition of FIR filter $H(z)$
- The structure is used to change filtering and down-sampling to down-sampling and filtering
- The number of operations remains the same but the filter operates at lower frequency
Type 1 polyphase decomposition

- Transpose of the polyphase decomposition of FIR filter $H(z)$
- The structure is used to change up-sampling and filtering into filtering and up-sampling
Type 2 polyphase decomposition

- Obtained by setting $R_i(z^M) = E_{M-i}(z^M)$
- In case of fractional sampling rate change, polyphase decomposition can be used to filter at rate $F_s/M$ instead of $LF_s$ where $F_s$ refers to the original sampling rate
Computationally efficient decimator
Computationally efficient interpolator

Type I

Type II
Commutator representation of interpolation and decimation with polyphase structure

interpolation  decimation
Polyphase fractional sampling/fractional delay filter

- Polyphase structure for P/Q fractional sampling
- Stage r provides a delay equal to r/P of the input sampling interval.
- Number of stages sets the resolution.

Commutator steps through branches with the increments of Q.
Polyphase fractional sampling/fractional delay filter

- Suppose that we want to calculate the output in the place \( r+d \) (\( r+d \) needn’t be rational any more) between the stages \( r \) and \( r+1 \).
- Linear interpolation of filter outputs between the nearest neighbors can be interpreted as interpolation of filter coefficients.

\[
y(n + \frac{r+d}{P}) = y(n + \frac{r}{P})(1 - d) + y(n + \frac{r+1}{P}) \cdot d
\]
\[
= (1 - d) \sum_{k} x(n - k)h_r(k) + d \sum_{k} x(n - k)h_{r+1}(k)
\]
\[
= \sum_{k} x(n - k)((1 - d)h_r(k) + d \cdot h_{r+1}(k))
\]
Cascaded Integrator Comb Filters

Linear-phase digital FIR filters without multipliers

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The simplest low-pass FIR filter is the N-point moving average filter or (excluding the scaling by 1/N), whose transfer function is given by

\[ H(z) = 1 + z^{-1} + z^{-2} + \cdots z^{-N-1} \]

Another form of \( H(z) \), known as recursive running-sum filter or boxcar filter, is given by

\[ H(z) = \frac{1 - z^{-N}}{1 - z^{-1}} \]
CIC decimator

- A realization of a factor-of-N decimator of the running sum filter (applying noble identity) is given below

- Because a decimator implemented by one running-sum filter usually does not provide enough stop-band attenuation, the filters are cascaded giving rise to *cascaded integrator comb* (CIC) filter
CIC Applications

- Down-sampling of the output of sigma-delta analog-to-digital converter
- Typically 16:1 down-sampling CIC filter followed by a 4-path polyphase filter or two half-band filters
1-4 stage CIC filters (without decimation)
CIC decimator

- Filter length and down-sampling factor needn’t be equal
- The structure of a two-stage CIC decimator is shown below

- It can be easily shown that the structure corresponds to a factor-of-R decimator with a length-RN running sum filter
CIC decimator

- Further flexibility in the design is obtained by including $K$ feedback paths before and $K$ feedforward paths after the down-sampler.

- Typically, the number of sections is 3-5.
- Increasing word length is a problem in practice.
CIC decimator

- The corresponding transfer function becomes

\[ H(z) = \left( \frac{1 - z^{-RN}}{1 - z^{-1}} \right)^{-K} \]

- The parameters N and K can be adjusted for a given down-sampling factor R to yield the desired out-of-band attenuation.
- Adding more sections improves out-of-band attenuation but also distorts the passband.
CIC Interpolator

- CIC interpolators are usually used in the last section of the multistage interpolator where the signal is sampled already in a high rate.

- The structure actually repeats input samples $x(k)$ $R$ times (when starting from zero state)
CIC interpolator

- This feature can be used to reduce complexity in multistage CIC interpolator
- Hold interpolator removes the comb section that requires the most number of bits – and integrator that requires the least number of bits