Exercise 3 – Variance reduction and Experimental design

3.1 (Problem 11.4., p.617 in the book by Law and Kelton)

Consider the following queueing model. Customers arrive according to a (stationary) Poisson process at rate 1 per minute. Server 1 provides service whose duration is exponentially distributed with a mean of 0.7 minutes. After leaving server 1, customers leave the system with probability p and go to server 2 with probability 1-p. At server 2, the duration of service is exponentially distributed with a mean 0.9 minutes. The service capacity for both servers is 1. The queueing discipline at both servers is first in first out (FIFO).

The system is initially empty and idle. It runs until 100 customers have been completely served. The total queueing delay for a customer that has to visit both servers is, naturally, the sum of delays at both places. The performance measure of interest is the expected average total waiting time in both of the queues.

a) Suppose there are 2 configurations of the system with p being either 0.3 or 0.8. Make 10 replications of the simulation with both values of p by first using independent random numbers. Then repeat the experiment using common random numbers (CRN) for both inter-arrival times and service times across the systems. Compare the estimated variances of the difference between the performance measures.

b) For p=0.3, make five pairs of runs using both independent sampling and antithetic variates within each pair. Compare the estimated variances of the performance measures.

Maintain proper synchronization of random numbers both in CRN and antithetic variates.

3.2 (Problem 12.1., p.666 in the book by Law and Kelton)

A manufacturing system contains m=5 machines, each subject to randomly occurring breakdowns. A machine runs for an amount of time that is exponentially distributed with a mean of 8 hours before breaking down. There are s repairmen to fix the broken machines, and it takes one repairman an exponential time with mean 2 hours to complete a repair of one machine. No more than one repairman can be assigned to work on a broken machine. If more than s=2 machines are broken down at a given time, they form a repair queue with FIFO discipline. Assume that it costs the system 50$ for each hour per each machine that is broken down. Further, it costs 10$ to employ each repairman (regardless of whether they are actually working or not). The system runs for exactly 800 hours. All machines are initially in working order.

Consider the following changes that could be made to lower the average cost per hour for the above described system:

a) The number of repairmen is increased to 4.
b) All machines are replaced with more reliable ones. The time to failure is 16 hours on the average. However, the cost of a failed machine is 100$ per hour.
c) Expert repairmen replace all standard ones. These men cost 15$ an hour, but can repair a machine in 1.5 hours on the average.

Perform a full 2^3 factorial experiment to examine the effects of the changes on the average hourly cost. Replicate the design 5 times to compute confidence intervals for all main and interaction effects.

3.3 (Demo)

Consider the model you built in exercise 3.1. The following changes to the system are made. The second server is dropped. After being served, the customers leave the system with probability p and re-enter the same server with probability $1-p$. However, the maximum number of times a given customer has to go through the service is 2. (In other words, each customer may be returned for re-processing only once.) The service time distribution and the queueing discipline remain the same.

We are interested in estimating the expected average total delay in queue for the first 100 customers that enter the system. For $p=0.8$, calculate the controlled estimate of the performance measure by using the total service time as a control variate. Use $n=10$ as the number of replications. Further, repeat the whole experiment 10 times and compare the variance in uncontrolled and controlled estimates for queueing delay.

Let us denote

X	the average total queueing time of a customer
Y	the average total service time of a customer
$\mu=\mathrm{E}(X)$	
$\nu=\mathrm{E}(Y)$	

Having observed (through simulation of n independent replications) the average queueing times $X_1, X_2, \ldots, X_n$ and the average service times $Y_1, Y_2, \ldots, Y_n$ the controlled estimate for μ is

$$\bar{X}_C^* = \bar{X} - \hat{a}^*(\bar{Y} - \nu), \text{ where}$$

$$\bar{X} = \sum_{j=1}^{n} X_j \text{ and } \bar{Y} = \sum_{j=1}^{n} Y_j$$

$$\hat{C}_{XY} = \frac{\sum_{j=1}^{n} (X_j - \bar{X})(Y_j - \bar{Y})}{n-1} \text{ and } S_Y^2 = \frac{\sum_{j=1}^{n} (Y_j - \bar{Y})^2}{n-1}$$

$$\hat{a}^* = \frac{\hat{C}_{XY}}{S_Y^2}$$

Naturally, we are aware that the actual value of $v = \mathrm{E}(Y)$=1.2*0.7=0.84 minutes, since 20 percent of the customers are sent back for re-processing.