Lecture 6: Particle Filtering

Simo Särkkä

February 26, 2020

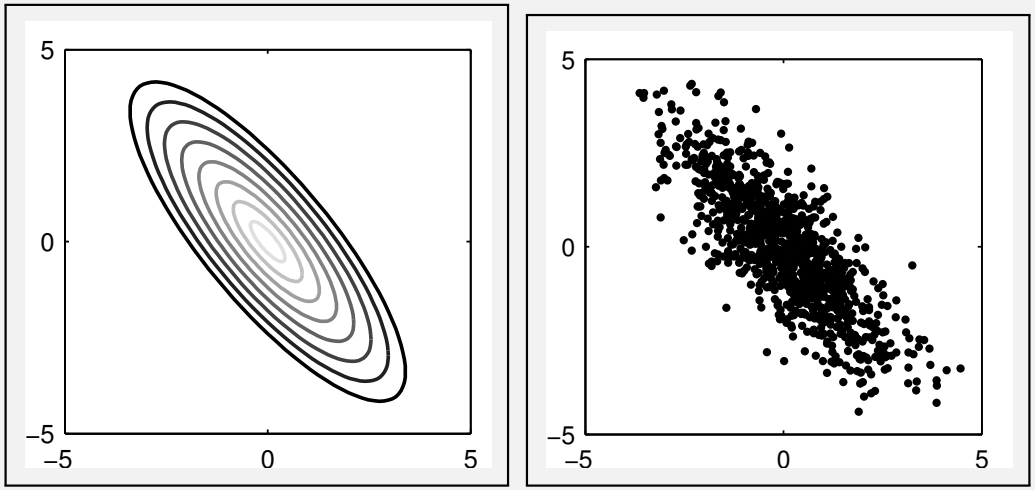
Learning Outcomes

1. Summary of the Last Lecture
2. Principle of Particle Filter
3. Monte Carlo Integration and Importance Sampling
4. Sequential Importance Sampling and Resampling
5. Particle Filter Properties
6. Summary and Demonstration

Summary of the Last Lecture

- Unscented transform (UT) approximates transformations of Gaussian variables by propagating sigma points through the non-linearity.
- In UT the mean and covariance are approximated as linear combination of the sigma points.
- The unscented Kalman filter uses unscented transform for computing the approximate means and covariance in non-linear filtering problems.
- A non-linear transformation can also be approximated with Gaussian moment matching.
- Gaussian filter is based on matching the moments with numerical integration $\Rightarrow$ many kinds of Kalman filters.
- Gauss-Hermite Kalman filter (GHKF) and Cubature Kalman filter (CKF) are examples of them.

Particle Filtering: Principle



- Particle filter uses Monte Carlo approximation instead of Gaussian approximation of the filtering distribution.
- More specifically, particle filter uses importance sampling for propagating the Monte Carlo samples in time.
- Animation: Kalman vs. Particle Filtering:
  - ▸ Kalman filter animation
  - ▸ Particle filter animation

Particle Filtering: Principle (cont.)

- Mathematically, particle filter forms a weighted sample (or particle) presentation $(\mathbf{x}^{(i)}, w^{(i)})$ of the filtering distribution:

$$p(\mathbf{x}) \approx \sum_{i=1}^{N} w^{(i)}\, \delta(\mathbf{x} - \mathbf{x}^{(i)}).$$

- E.g., mean and covariance can then be approximated as

$$\mathrm{E}[\mathbf{x}] \approx \sum_{i=1}^{N} w^{(i)}\, \mathbf{x}^{(i)} = \mathbf{m}_{PF}$$

$$\mathrm{Cov}[\mathbf{x}] \approx \sum_{i=1}^{N} w^{(i)}\, (\mathbf{x}^{(i)} - \mathbf{m}_{PF})\, (\mathbf{x}^{(i)} - \mathbf{m}_{PF})^{\top}$$

- Other statistics can be approximated analogously.
- Particle filter samples from the Bayesian filtering equations with sequential importance sampling.

Monte Carlo Integration

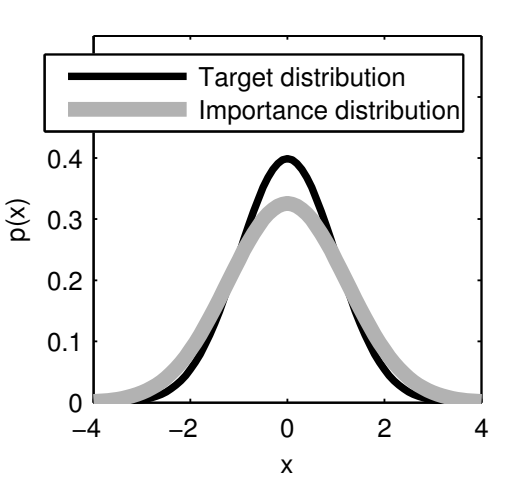
- In Bayesian inference we often want to compute posterior expectations of the form

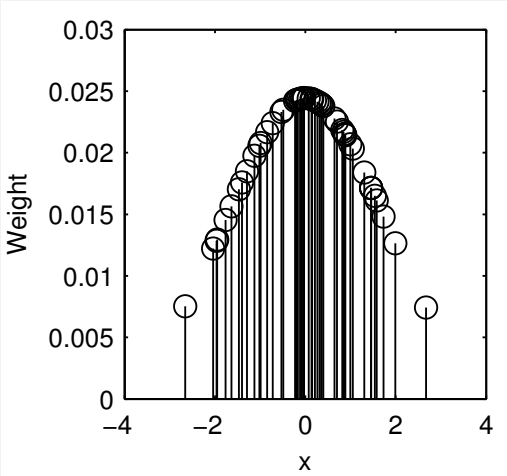
$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \,|\, \mathbf{y}_{1:T}] = \int \mathbf{g}(\mathbf{x})\, p(\mathbf{x} \,|\, \mathbf{y}_{1:T})\, \mathrm{d}\mathbf{x}$$

- For example, posterior mean and posterior covariance are such expectations.
- Monte Carlo: draw N independent random samples from $\mathbf{x}^{(i)} \sim p(\mathbf{x} \,|\, \mathbf{y}_{1:T})$ and estimate the expectation as

$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \,|\, \mathbf{y}_{1:T}] \approx \frac{1}{N} \sum_{i=1}^{N} \mathbf{g}(\mathbf{x}^{(i)}).$$

Importance Sampling: Basic Version [1/2]





- In practice, we rarely can directly draw samples from the distribution $p(\mathbf{x} \mid \mathbf{y}_{1:T})$.
- In importance sampling (IS), we draw samples from an importance distribution $\mathbf{x}^{(i)} \sim \pi(\mathbf{x} \mid \mathbf{y}_{1:T})$ and compute weights $\tilde{w}^{(i)}$ such that

$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \mid \mathbf{y}_{1:T}] \approx \sum_{i=1}^{N} \tilde{w}^{(i)} \, \mathbf{g}(\mathbf{x}^{(i)})$$

Importance Sampling: Basic Version [2/2]

- Importance sampling is based on the identity

$$\begin{aligned} \mathrm{E}[\mathbf{g}(\mathbf{x}) \,|\, \mathbf{y}_{1:T}] &= \int \mathbf{g}(\mathbf{x})\, p(\mathbf{x} \,|\, \mathbf{y}_{1:T})\, \mathrm{d}\mathbf{x} \\ &= \int \left[\mathbf{g}(\mathbf{x}) \, \frac{p(\mathbf{x} \,|\, \mathbf{y}_{1:T})}{\pi(\mathbf{x} \,|\, \mathbf{y}_{1:T})} \right] \pi(\mathbf{x} \,|\, \mathbf{y}_{1:T})\, \mathrm{d}\mathbf{x} \end{aligned}$$

- Thus we can form a Monte Carlo approximation as follows:

$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \,|\, \mathbf{y}_{1:T}] \approx \frac{1}{N} \sum_{i=1}^{N} \frac{p(\mathbf{x}^{(i)} \,|\, \mathbf{y}_{1:T})}{\pi(\mathbf{x}^{(i)} \,|\, \mathbf{y}_{1:T})} \, \mathbf{g}(\mathbf{x}^{(i)})$$

- That is, the importance weights can be defined as

$$\tilde{w}^{(i)} = \frac{1}{N} \, \frac{p(\mathbf{x}^{(i)} \,|\, \mathbf{y}_{1:T})}{\pi(\mathbf{x}^{(i)} \,|\, \mathbf{y}_{1:T})}$$

Importance Sampling: Weight Normalization

- The problem is that we need to evaluate the normalization constant of $p(\mathbf{x}^{(i)} \mid \mathbf{y}_{1:T})$ – often not possible.
- However, it turns out that we get a valid algorithm if we define unnormalized importance weights as

$$w^{*(i)} = \frac{p(\mathbf{y}_{1:T} \mid \mathbf{x}^{(i)})\, p(\mathbf{x}^{(i)})}{\pi(\mathbf{x}^{(i)} \mid \mathbf{y}_{1:T})}$$

  and then normalize them:

$$w^{(i)} = \frac{w^{*(i)}}{\sum_j w^{*(j)}}$$

- The (weight-normalized) importance sampling approximation is then

$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \mid \mathbf{y}_{1:T}] \approx \sum_{i=1}^{N} w^{(i)}\, \mathbf{g}(\mathbf{x}^{(i)})$$

Importance Sampling: Algorithm

Importance Sampling

- Draw N samples from the importance distribution:

$$\mathbf{x}^{(i)} \sim \pi(\mathbf{x} \,|\, \mathbf{y}_{1:T}), \qquad i = 1, \ldots, N.$$

- Compute the unnormalized weights by

$$w^{*(i)} = \frac{p(\mathbf{y}_{1:T} \,|\, \mathbf{x}^{(i)})\, p(\mathbf{x}^{(i)})}{\pi(\mathbf{x}^{(i)} \,|\, \mathbf{y}_{1:T})},$$

  and the normalized weights by

$$w^{(i)} = \frac{w^{*(i)}}{\sum_{j=1}^{N} w^{*(j)}}.$$

Importance Sampling: Properties

- The approximation to the posterior expectation of $\mathbf{g}(\mathbf{x})$ is

$$\mathrm{E}[\mathbf{g}(\mathbf{x}) \mid \mathbf{y}_{1:T}] \approx \sum_{i=1}^{N} w^{(i)}\, \mathbf{g}(\mathbf{x}^{(i)}).$$

- The posterior probability density approximation can be formally written as

$$p(\mathbf{x} \mid \mathbf{y}_{1:T}) \approx \sum_{i=1}^{N} w^{(i)}\, \delta(\mathbf{x} - \mathbf{x}^{(i)}),$$

  where $\delta(\cdot)$ is the Dirac delta function.
- The efficiency depends on the choice of the importance distribution.

Sequential Importance Sampling: Idea

- Sequential Importance Sampling (SIS) is concerned with models

$$\mathbf{x}_k \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$$
$$\mathbf{y}_k \sim p(\mathbf{y}_k \mid \mathbf{x}_k)$$

- The SIS algorithm uses a weighted set of particles $\{(w_k^{(i)}, \mathbf{x}_k^{(i)}) \ : \ i = 1, \ldots, N\}$ such that

$$\mathrm{E}[\mathbf{g}(\mathbf{x}_k) \mid \mathbf{y}_{1:k}] \approx \sum_{i=1}^{N} w_k^{(i)} \mathbf{g}(\mathbf{x}_k^{(i)}).$$

- Or equivalently

$$p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) \approx \sum_{i=1}^{N} w_k^{(i)} \delta(\mathbf{x}_k - \mathbf{x}_k^{(i)}),$$

  where $\delta(\cdot)$ is the Dirac delta function.
- Uses importance sampling sequentially.

Sequential Importance Sampling: Derivation [1/2]

- Let's consider the full posterior distribution of states $\mathbf{x}_{0:k}$ given the measurements $\mathbf{y}_{1:k}$.
- We get the following recursion for the posterior distribution:

$$\begin{aligned} p(\mathbf{x}_{0:k} \,|\, \mathbf{y}_{1:k}) &\propto p(\mathbf{y}_k \,|\, \mathbf{x}_{0:k}, \mathbf{y}_{1:k-1})\, p(\mathbf{x}_{0:k} \,|\, \mathbf{y}_{1:k-1}) \\ &= p(\mathbf{y}_k \,|\, \mathbf{x}_k)\, p(\mathbf{x}_k \,|\, \mathbf{x}_{0:k-1}, \mathbf{y}_{1:k-1})\, p(\mathbf{x}_{0:k-1} \,|\, \mathbf{y}_{1:k-1}) \\ &= p(\mathbf{y}_k \,|\, \mathbf{x}_k)\, p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1})\, p(\mathbf{x}_{0:k-1} \,|\, \mathbf{y}_{1:k-1}). \end{aligned}$$

- We could now construct an importance distribution $\mathbf{x}_{0:k}^{(i)} \sim \pi(\mathbf{x}_{0:k} \,|\, \mathbf{y}_{1:k})$ and compute the corresponding (normalized) importance weights as

$$w_k^{(i)} \propto \frac{p(\mathbf{y}_k \,|\, \mathbf{x}_k^{(i)})\, p(\mathbf{x}_k^{(i)} \,|\, \mathbf{x}_{k-1}^{(i)})\, p(\mathbf{x}_{0:k-1}^{(i)} \,|\, \mathbf{y}_{1:k-1})}{\pi(\mathbf{x}_{0:k}^{(i)} \,|\, \mathbf{y}_{1:k})}.$$

Sequential Importance Sampling: Derivation [2/2]

- Let's form the importance distribution recursively as follows:

$$\pi(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) = \pi(\mathbf{x}_k \mid \mathbf{x}_{0:k-1}, \mathbf{y}_{1:k})\, \pi(\mathbf{x}_{0:k-1} \mid \mathbf{y}_{1:k-1})$$

- Expression for the importance weights can be written as

$$w_k^{(i)} \propto \frac{p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)})\, p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})} \underbrace{\frac{p(\mathbf{x}_{0:k-1}^{(i)} \mid \mathbf{y}_{1:k-1})}{\pi(\mathbf{x}_{0:k-1}^{(i)} \mid \mathbf{y}_{1:k-1})}}_{\propto w_{k-1}^{(i)}}$$

- Thus the weights satisfy the recursion

$$w_k^{(i)} \propto \frac{p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)})\, p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})}\, w_{k-1}^{(i)}$$

Sequential Importance Sampling: Algorithm

Sequential Importance Sampling

- **Initialization:** Draw N samples $\mathbf{x}_0^{(i)}$ from the prior

$$\mathbf{x}_0^{(i)} \sim p(\mathbf{x}_0)$$

  and set $w_0^{(i)} = 1/N$.

- **Prediction:** Draw N new samples $\mathbf{x}_k^{(i)}$ from importance distributions

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_k \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})$$

- **Update:** Calculate new weights according to

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)})\; p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})}$$

Sequential Importance Sampling: Degeneracy

- The problem in SIS is that the algorithm is degenerate
- It can be shown that the variance of the weights increases at every step
- It means that we will always converge to single non-zero weight $w^{(i)} = 1$ and the rest being zero – not very useful algorithm.
- Solution: resampling!

Sequential Importance Resampling: Resampling Step

- Sequential Importance Resampling (SIR) algorithm adds the following resampling step to SIS algorithm:

Resampling

- Interpret each weight $w_k^{(i)}$ as the probability of obtaining the sample index i in the set $\{\mathbf{x}_k^{(i)} \mid i = 1, \ldots, N\}$.
- Draw N samples from that discrete distribution and replace the old sample set with this new one.
- Set all weights to the constant value $w_k^{(i)} = 1/N$.

- There are many algorithms for implementing this – stratified resampling is optimal in terms of variance.

Sequential Importance Resampling: Effective Number of Particles

- A simple way to do resampling is at every step – but every resampling operation increases variance.
- We can also resample at, say, every Kth step.
- In adaptive resampling, we resample when the effective number of samples is too low (say, $N/10$):

$$n_{\text{eff}} \approx \frac{1}{\sum_{i=1}^{N} \left(w_k^{(i)} \right)^2},$$

Sequential Importance Resampling: Algorithm

Sequential Importance Resampling

- Draw samples $\mathbf{x}_k^{(i)}$ from the importance distribution:

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_k \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k}), \qquad i = 1, \ldots, N.$$

- Calculate new weights

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)})\, p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})}{\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k})}, \qquad i = 1, \ldots, N,$$

  and normalize them to sum to unity.
- If the effective number of particles is too low, perform resampling.

Sequential Importance Resampling: Bootstrap filter

- In bootstrap filter we use the dynamic model as the importance distribution

$$\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k}) = p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)})$$

and resample at every step:

Bootstrap Filter

- Draw samples $\mathbf{x}_k^{(i)}$ from the dynamic model:

$$\mathbf{x}_k^{(i)} \sim p(\mathbf{x}_k \mid \mathbf{x}_{k-1}^{(i)}), \qquad i = 1, \ldots, N.$$

- Calculate new weights

$$w_k^{(i)} \propto p(\mathbf{y}_k \mid \mathbf{x}_k^{(i)}), \qquad i = 1, \ldots, N,$$

and normalize them to sum to unity.

- Perform resampling.

Sequential Importance Resampling: Optimal Importace Distribution

- The optimal importance distribution is

$$\pi(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{0:k-1}^{(i)}, \mathbf{y}_{1:k}) = p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k)$$

- Then the weight update reduces to

$$w_k^{(i)} \propto w_{k-1}^{(i)}\, p(\mathbf{y}_k \mid \mathbf{x}_{k-1}^{(i)}), \qquad i = 1, \ldots, N.$$

- The optimal importance distribution can be used, for example, when the state space is finite.

Sequential Importance Resampling: Importace Distribution via Kalman Filtering

- We can also form a Gaussian approximation to the optimal importance distribution:

$$p(\mathbf{x}_k^{(i)} \mid \mathbf{x}_{k-1}^{(i)}, \mathbf{y}_k) \approx \mathrm{N}(\mathbf{x}_k^{(i)} \mid \tilde{\mathbf{m}}_k^{(i)}, \tilde{\mathbf{P}}_k^{(i)}).$$

  by using a single prediction and update steps of a Gaussian filter starting from a singular distribution at $\mathbf{x}_{k-1}^{(i)}$.
- We can also replace above with the result of a Gaussian filter $\mathrm{N}(\mathbf{m}_{k-1}^{(i)}, \mathbf{P}_{k-1}^{(i)})$ started from a random initial mean.
- A very common way seems to be to use the previous sample as the mean: $\mathrm{N}(\mathbf{x}_{k-1}^{(i)}, \mathbf{P}_{k-1}^{(i)})$.
- A particle filter with UKF proposal has been given name unscented particle filter (UPF) – you can invent new PFs easily this way.

Particle Filter: Advantages

- No restrictions in model – can be applied to non-Gaussian models, hierarchical models etc.
- Global approximation.
- Approaches the exact solution, when the number of samples goes to infinity.
- In its basic form, very easy to implement.

Particle Filter: Disadvantages

- Computational requirements much higher than of the Kalman filters.
- Problems with nearly noise-free models, especially with accurate dynamic models.
- Good importance distributions quite tricky to implement.
- Very hard to find programming errors (i.e., to debug).

Summary

- Particle filters can be used for approximate filtering in general probabilistic state-space models.
- Particle filters use weighted set of samples (particles) for approximating the filtering distributions.
- Sequential importance resampling (SIR) is the general framework and bootstrap filter is a simple special case of it.
- EKF, UKF and other Gaussian filters can be used for forming good importance distributions.
- The optimal importance distribution is the minimum variance importance distribution.

Particle Filter: Demo

- The discretized pendulum model:

$$\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k-1}^1 + x_{k-1}^2 \, \Delta t \\ x_{k-1}^2 - g \, \sin(x_{k-1}^1) \, \Delta t \end{pmatrix}}_{\mathbf{f}(\mathbf{x_{k-1}})} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}$$

$$y_k = \underbrace{\sin(x_k^1)}_{\mathbf{h}(\mathbf{x_k})} + r_k,$$

- $\Rightarrow$ *Matlab demonstration*