Aalto University School of Science
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MS-E1281 Real analysis, spring 2020

Homework assignment 3

Topic: The Hardy-Littlewood maximal function.

Deadline 23 March 2020 at 16:00.

1. Assume that $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Prove the followng claims:
   (a) (Sublinearity) $M(f+g)(x) \le Mf(x) + Mg(x)$.
   (b) (Homogeneity) $M(af)(x) = |a|Mf(x)$, $a \in \mathbb{R}$.
   (c) (Translation invariance) $M(\tau_y f)(x) = (\tau_y Mf)(x)$, $y \in \mathbb{R}^n$, where $\tau_y f(x) = f(x+y)$.
   (d) (Scaling invariance) $M(\delta_a f)(x) = (\delta_a Mf)(x)$, where $\delta_a f(x) = f(ax)$ with $a > 0$.

2. Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show that for each fixed $0 < r < \infty$, the mapping
$$x \mapsto \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy, \quad x \in \mathbb{R}^n,$$
is continuous in $\mathbb{R}^n$. How can this fact be used to conclude that the Hardy-Littlewood maximal function is lower semicontinuous?

3. Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show that for each fixed $x \in \mathbb{R}^n$ the mapping
$$r \mapsto \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy, \quad 0 < r < \infty,$$
is continuous in $\mathbb{R}_+$.

4. Show that if $Mf(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, then $Mf(x) = 0$ for every $x \in \mathbb{R}^n$.

5. Show that if $Mf(x_0) < \infty$ for some $x_0 \in \mathbb{R}^n$, then $Mf(x) < \infty$ for almost every $x \in \mathbb{R}^n$.

6. Let $r > 0$. Show that there are constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that
$$\frac{c_1 r^n}{(|x|+r)^n} \le M(\chi_{B(0,r)})(x) \le \frac{c_2 r^n}{(|x|+r)^n}$$
for every $x \in \mathbb{R}^n$. How can this be used to show that the Hardy-Littlewood maximal operator does not map $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?