Aalto University School of Science
Department of Mathematics and Systems Analysis Kinnunen

MS-E1281 Real analysis, spring 2020

Homework assignment 6

Topic: Duals and weak convergence.

Deadline 20 April 2020 at 16:00.

1. Let μ and ν be finite Radon measures on $\mathbb{R}^n$.
   (a) If $\nu \ll \mu$ and f is a nonnegative μ-measurable function, show that
   $$\int_A f\,d\nu = \int_A f D_\mu \nu\,d\mu$$
   for every Borel set $A \subset \mathbb{R}^n$.
   (b) Prove the corresponding claim for $f \in L^1(\mathbb{R}^n;\nu)$.
2. Assume that $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$.
   (a) Show that
   $$\left|\int_{\mathbb{R}^n} fg\,dx\right| \le \|f\|_p$$
   for every $g \in L^{p'}(\mathbb{R}^n)$ with $\|g\|_{p'} \le 1$.
   (b) Show that
   $$\|f\|_p = \sup\left\{\left|\int_{\mathbb{R}^n} fg\,dx\right| : \|g\|_{p'} \le 1\right\}.$$
   Hint:
   $$g = \frac{|f|^{p/p'}\operatorname{sgn} f}{\|f\|_p^{p/p'}}.$$
3. Assume that $f \in L^1(\mathbb{R}^n)$.
   (a) Show that
   $$\left|\int_{\mathbb{R}^n} fg\,dx\right| \le \|f\|_1$$
   for every $g \in L^\infty(\mathbb{R}^n)$ with $\|g\|_\infty \le 1$.
   (b) Show that
   $$\|f\|_1 = \sup\left\{\left|\int_{\mathbb{R}^n} fg\,dx\right| : \|g\|_\infty \le 1\right\}.$$

4. Let $\mu_i = \delta_{1+1/i}$, $i = 1, 2, \ldots$, be Dirac's measures on $\mathbb{R}$.

   (a) Show that μ_i converges weakly as $i \to \infty$ and determine the weak limit μ.

   (b) Give an an example of a Borel set $A \subset \mathbb{R}$ such that

$$\lim_{i\to\infty} \mu_i(A) \neq \mu(A).$$

5. Let $f_i : (0, 2\pi) \to \mathbb{R}$, $f_i(x) = \sin(ix)$, $i = 1, 2, \ldots$.

   (a) Show that f_i converges weakly to 0 in $L^p((0, 2\pi))$.

   (b) Show that f_i does not converge to 0 in $L^p((0, 2\pi))$.

6. Assume that μ_i, $i = 1, 2, \ldots$, are Radon measures on $\mathbb{R}^n$ with $\mu_i \rightharpoonup \mu$ as $i \to \infty$. Show that

$$\lim_{i\to\infty} \mu_i(A) = \mu(A)$$

   for every bounded Borel set $A \subset \mathbb{R}^n$ with $\mu(\partial A) = 0$.