Mathematics for Economists: Lecture 1

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Spring 2020

Welcome to the course

- Course logistics
  - Lectures Mon, Wed at 13:15 - 14:45
  - Review sessions with Amin Mohazab Thu 10:15 - 11:45
  - Weekly Problem Sets to be returned via Mycourses on specified due date
  - 20% of the grade based on problem sets, 80% on final exam
  - To succeed in the course, you should attempt all problem sets
  - Exam May 25, 9:00-12:00

Contents I

- Course contents: Part I
  - Lecture 0: Prerequisites from Linear Algebra
    Readings: Handout on Basic Linear Algebra, S&B: Chapters 7, 8.1 - 8.4
  - Lecture 1: Introduction and the language of mathematics
    Readings: Sections 1-2 in the notes, S&B: Chapter 13
  - Lectures 2-4: Multivariate Calculus Readings: Sections 3-4 in the notes, S&B: Chapters 14, 15
  - Lecture 5: Unconstrained Optimization Readings: Section 5 in the notes, S&B: Chapters 16, 17
  - Lecture 6: Convexity and Concavity Readings: Section 6 in the notes, S&B: Chapter 21

Contents II

- Course contents: Part II
  - Lecture 7-8: Constrained Optimization
    Readings: S&B Chapters 18, 19
  - Lecture 9-10: Economic Applications of Constrained Optimization
    Readings: S&B: Chapters 20, 22
  - Lectures 11-12: Linear Dynamical Systems
    Readings: S&B: Chapters 23, 25.1, 25.2

Economic models

- Economics studies the allocation of scarce resources amongst competing ends
  - what are the ways to allocate?
  - how to evaluate the results?
  - what do we mean by scarcity?
  - how can we formalize such questions?
- Individualistic approach: economic agents are autonomous decision makers
- They act in pursuit of individual objectives or goals
  - agents do not make systematic mistakes in their choices
  - they act within constraints
  - they react to changes on their environment

Economic models

- Equilibrium analysis to guarantee the consistency of individual decisions
  - in competitive markets: equilibrium brought about by price mechanism
  - in games: equilibrium from consistency of expectations and realized behavior

Optimizing agents

- Economic agents (also called decision makers) have objectives summarized in their objective functions
  - utility function of a consumer
  - profit function of a firm
  - total surplus for an economic planner
- Autonomous decisions:
  - each agent chooses her own actions
- Decisions in line with the objectives
  - each agent maximizes her objective function
- Economic choices constrained by scarcity

Endogenous and exogenous variables

- Economic models are chosen by the modeler
- Idea is to pick the most important features of an economic situation and ignore the rest
- Every model has variables that are determined within the model
  - endogenous variables
- Interesting models have variables not determined within the model
  - exogenous variables
- Exogenous variables and parameters of the model are similar in nature

Mathematical formulation

- In order to formulate the problem, we need the following ingredients:
  - Choices x from the set of choice variables: X
  - Evaluation of choices: objective function $f : X \to \mathbb{R}$
  - Scarcity in the form of feasible set: $F = \{x | g(x) \leq 0\}$
  - Possible parameters and other (exogenous) variables, $\{\alpha, \beta, ...\}$ to include in f, g
  - Exogenous variables are not determined in the model
- For concreteness, let's consider some economic problems from Principles of Economics I

Examples

- Consumer choice between food and leisure
  - x_1 food consumption, x_2 leisure.
    $X = \{(x_1, x_2) | x_i \in \mathbb{R},$ for $i \in \{1, 2\}, x_1 \geq 0, 0 \leq x_2 \leq 24\}$.
  - Utility from x: $f(x; \alpha) = f(x_1, x_2; \alpha)$, where α is a preference parameter
  - Feasible set: $p_1 x_1 \leq w(24 - x_2)$
  - Price of food p_1 and wages w are exogenous variables
  - Exercise: Write the constraint in form $g(x_1, x_2) \leq 0$.
- Best responses of player 1 in two player games:
  - Own action $x_1 \in X_1$ (row in the matrix)
  - Payoff from own action: $f(x_1; x_2)$, where x_2 is the exogenous variable (for best responses of 1, we just compute the payoff for all possible choices $x_2 \in X_2$, i.e. for all rows in the matrix)
  - In this context, no further feasibility constraint
  - Of course when solving the game, x_2 becomes also an endogenous variable.

General form:

- In general, we have the problem

$$\max_{x \in X} f(x;\alpha)$$

$$\text{subject to } g(x;\beta) \leq 0.$$

- What is a solution to the problem? An x^* such that
  - i)

    $$g(x^*;\beta) \leq 0.$$

    i.e. the solution is feasible.
  - ii)

    $$f(x^*;\alpha) \geq f(y;\alpha),$$

    for all y such that:

    $$g(y;\beta) \leq 0.$$

- In other words, optimal choice attains the highest value of the objective function within the feasible set

Mathematical structure:

- What kinds of variables are x, α, β?
  - most often real numbers, real vectors or sometimes discrete choices (such as choosing the row in a matrix or choosing between a red and a blue car)
- When does a solution exist?
  - Weierstrass theorem (you will see this in part II of the course) or other existence results (to be just hinted at)
- How to find a solution?
  - one of the main questions for this course
  - usually with the help of calculus
  - calculus is not of much help for discrete problems, in more advanced courses tools for handling this to some extent

Mathematical structure:

- Is the solution unique?
  - concavity and convexity of the objective function key for this
- How do endogenous variables react to changes in exogenous variables?
  - comparative statics
  - implicit function theorem is the key tool for this and one of our first goals in this course

Functions and linear functions

- A function f associates to each of the points in its domain X an element in its co-domain Y.
- We write $y = f(x) \in Y$ and also:

$$f : X \to Y.$$

- Functions are sometimes called mappings, maps or transformations. The sets X, Y can be very general and the function can take many forms. Here are some examples:
  - X is the set of all humans, dead and alive and $Y = X$. For each x, $f(x)$ is the mother of x.
  - X is the population of Finland, $Y = \{0, 1\}$. $f(x) = 1$ if x has Covid -antibodies in her blood and $f(x) = 0$ otherwise.
  - X is the set of feasible portfolios, $Y = \mathbb{R}$. $f(x)$ is the expected return on portfolio x.
  - X is the set of all humans, dead and alive and $Y = X$. The following relation $y = f(x)$ if y is a child of x is not a function since some x have many children and some have none.

Functions and linear functions

- Our goal in this part of the course is to gain an understanding of real-valued and vector-valued functions of multiple variables, i.e. functions

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

1. Preferences of a consumer expressed through a utility function defined on consumption vectors
2. Profit of a firm as a function of the vector of outputs and the inputs chosen
3. Simultaneous equilibrium in many markets as the intersection of demand and supply functions
4. Likelihood function of observed data as a function of the parameters of the distribution of error terms

Functions and linear functions

- When are functions easy to understand?
- If the function can be extrapolated from a few representative cases.
- For vectors $x = (x_1, ..., x_n) \in \mathbb{R}^n$, scalar multiples of the vector by real number α given by $\alpha x = (\alpha x_1, ..., \alpha x_n) \in \mathbb{R}^n$.
- If we define the sum of two vectors $(x_1, ..., x_n) \in \mathbb{R}^n$ and $(y_1, ..., y_n) \in \mathbb{R}^n$ as $(x_1 + y_1, ..., x_n + y_n) \in \mathbb{R}^n$, i.e. by adding each coordinate, then $x + y$ is also a vector.
- We write for $x \in \mathbb{R}^n$:

$$x = (x_1, ..., x_n) = x_1\mathbf{e_1} + x_2\mathbf{e_2} + ... + x_n\mathbf{e_n}$$

$$= \sum_{i=1}^{n} x_i\mathbf{e_i},$$

where $\mathbf{e_i}$ is the i^{th} unit coordinate vector that has zeros at all but the i^{th} coordinate and 1 as the i^{th} coordinate.

Functions and linear functions

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ associating a vector $y = (y_1, .., y_m)$ to each $(x_1, ..., x_n)$ is called linear if for all $x, x' \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, i) $f(\alpha x) = \alpha f(x)$ and ii) $f(x + x') = f(x) + f(x')$.
- Here is the great thing about linear functions. If we know $f(\mathbf{e_i})$ for $i \in \{1, ..., n\}$, then we know $f(x)$ for all $x \in \mathbb{R}^n$.

$$f(x) = f(\sum_{i=1}^{n} x_i \mathbf{e_i}) = \sum_{i=1}^{n} x_i f(\mathbf{e_i}).$$

- Suppose now that $Y = \mathbb{R}^m$. Define $\mathbf{a_i} = f(\mathbf{e_i}) \in \mathbb{R}^m$. Then we can represent the linear function f by an $(m \times n)$ -matrix A:

$$A = [\mathbf{a_1}, ..., \mathbf{a_n}].$$

and

$$f(x) = Ax.$$

Functions and linear functions

- We say that a function $f : X \to Y$ is surjective or onto if for all $y \in Y$, there is (at least one) $x \in X$ such that $y = f(x)$.
- f is called injective or one-to-one if for each $y in Y$, there is at most one $x \in X$ such that $y = f(x)$. Finally, it is called bijective if it is both injective and surjective.
- For a bijective function, we can define an inverse function $f^{-1} : Y \to X$ so that we have for all $x \in X$ and all $y \in Y$:

$$x = f^{-1}(f(x)), \quad y = f(f^{-1}(y)).$$

Functions and linear functions

- Function f is injective if $rank(A) = n$. It is bijective if $m = rank(A) = n$. In this case, the inverse function f^{-1} is represented by the inverse matrix A^{-1}.
- If A^{-1} exists, then we can solve the linear system $y = Ax$ as $x = A^{-1}y$.
- For a linear function f represented by an $(m \times n$ -matrix A, we can use our results from the systems of equations to classify f. f is surjective if for all $y \in Y$, there is an $x \in X$ such that:

$$Ax = y.$$

- This is the case when $rank(A) = m$.

Non-linear functions

- Unfortunately linear functions do not fit well in all economic situations. A partial list of issues that limit the usefulness of linear models is:
  1. Diminishing marginal returns are not captured by linear models
  2. Only linear indifference curves are consistent with linear utility functions
  3. There is no reason why prices and quantities should have linear dependencies
- Need non-linear models

Utility function

- A consumer considers consuming k different goods
- A consumption plan is a positive vector $x \in \mathbb{R}^k_+$.
- A consumer has rational preferences if i) for all consumption plans x, y, she either prefers x to y or $y$to x or both, ii) for all consumption plans x, y, z, if she prefers x to y and y to z, then she prefers x to z.
- In a later course in microeconomics, you will see that (continuous) rational preferences can be represented by a utility function $u : \mathbb{R}^k_+ \to \mathbb{R}$
- This means that $u(x) = u(x_1, ...x_k) \geq u(y_1, ...y_k) = u(y)$ if and only if x is preferred to y
- In other words, the utility function is just a convenient summary of the preferences
- Non-linearity of u reflects non-constant marginal rates of substitution between goods, diminishing marginal utility, in models of uncertainty the risk attitudes etc.

Production function

- A firm produces output from two inputs, labor $L \in \mathbb{R}_+$ and capital $K \in \mathbb{R}_+$
- the output is given by $Y = f(K, L) \in \mathbb{R}_+$
- Non-linearity arises from diminishing marginal returns, diminishing diminishing or increasing returns to scale

Solving non-linear models

- The main problem in analyzing non-linear functions is that their shape varies sometimes greatly as x moves around in the domain X.
- Consider a linear function $f : \mathbb{R}^2 \to \mathbb{R}$. For all $\hat{x}_0 = (\hat{x}_1, \hat{x}_2)$, we have

$$f(\hat{x} + y) - f(\hat{x}) = f(\hat{x}_1 + y_1, \hat{x}_2 + y_2) - f(\hat{x}_1, \hat{x}_2)$$

$$= f(\hat{x}) + f(y_1, y_2) - f(\hat{x}) = f(y_1, y_2) = f(y).$$

- Moving in the direction y induces a change in the value of the function that is independent of the starting point of the movement $\hat{x}$.
- This is not true of non-linear functions. Must find other methods.

Solving non-linear models

Options:
i) Numerical methods. Graph the function using Matlab, R or some other language and find solutions to the model using numerical algorithms.
Not pursued in this course.
ii) Local methods Idea: A well behaved function is well approximated by a (different) linear function near any point in its domain.
Use the linear approximation to make inferences about the true non-linear function.
Analogy, the surface of the earth is curved, i.e. non-linear but for most everyday uses, a two-dimensional approximation (a map) is good enough.
The main tool for local analysis: differential calculus