Mathematics for Economists
ECON-C1000
Spring 2020
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Problem Set 1: Due April 22, 2020

1. Linear models:

   (a) Consider this simplest possible model of equilibrium in an industry. The demand is given by $q^d = a - bp$, the supply by $q^s = c + dp$. Market clearing for a closed economy states: $q^s = q^d$. Assume that $a, b, c, d > 0$ and $a > b$.

      i. After setting $q^s = q^d = q$ the equation system for market equilibrium determination in terms of p, q.

      ii. Changing a by δa changes the equilibrium price and quantity. Draw the figure depicting the equilibrium and consider this effect. Denote the changes by $\Delta q, \Delta p$. What is your guess for $\frac{\Delta q}{\Delta p}$?

      iii. Answer the same question but for changes in c.

      iv. Solve for these changes from the linear model.

      v. If you see only a single (p, q) pair, you cannot say much about supply or demand. You just have a single observation. What king of variation in data would you like to see if you want to find the supply curve? Give real-life examples of such variation.

   (b) A stochastic matrix is a matrix where the elements in each column sum up to one. In other words, if A is a stochastic $n \times n$ -matrix, then $\sum_{i=1}^{n} a_{ij} = 1$ for all j.

      i. Show that the matrix
      $$\begin{pmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{pmatrix}$$
      has full rank, but the matrix
      $$\begin{pmatrix} -0.7 & 0.5 & 0.4 \\ 0.4 & -0.9 & 0.5 \\ 0.3 & 0.4 & -0.9 \end{pmatrix}$$
      does not have full rank.

      ii. Show that if A is a stochastic matrix, then $I - A$ and $A - I$ do not have full rank, where I is the $n \times n$ identity matrix.

   (c) Consider a class of 30 students. A sociologist wants to understand the social hierarchy in the class and asks each student to endorse some other students in the class. Based on the responses she designs a ranking for the students. We want to see how this can be done using the tools of linear models.

i. Form a matrix of endorsements as follows: Identify each row and each column in a 30×30 -matrix A with a student. The students endorsed by i form the i^{th} column of the matrix as follows. If i endorses n_i other students, set $a_{ji} = \frac{1}{n_i}$ if i endorsed j and 0 otherwise (no self-endorsements). What can you say about $\sum_{i=1}^{30} a_{ji}$?

ii. What is the interpretation of $\sum_{i=1}^{30} a_{ij}$? Does this for a good way of ranking the importance of the students?

iii. Maybe endorsements from important students are more important for the ranking. To capture this idea, consider a ranking vector $x = (x_1, ..., x_{30})$ for the students. Require that the ranking of each student i is the sum of the endorsements a_{ij} wighted by the importance of the endorsing student j. Write the linear model capturing this idea:

$$Ax = x.$$

Show that $A-I$ does not have full rank. This means that we have a non-zero solution to the linear system.

iv. You can take it on faith (or find a proof using either Brouwer's fixed point theorem or Farkas' lemma) that a strictly positive solution exists. Normalize x so that $\sum_{i=1}^{30} x_i = 1$. Show by an example that there can be many such non-zero solutions if the students can be divided into cliques that do not endorse students in other cliques.

v. (No question here, but just for your information) If you perturb the matrix of endorsements A to be $A' = (1-\epsilon)A + \frac{\epsilon}{n}\mathbf{1}$, where $\mathbf{1}$ is a matrix whose elements are all equal to 1, then

$$A'x = x, \quad \sum_{i=1}^{30} x_i = 1$$

has a unique positive solution. This is one consequence of the famous Perron-Frobenius theorem.

(d) Consider next the set of all web pages and the replace the endorsements by outward links from page i to page j. The resulting matrix B may have columns without outward links. Replace any such column by a column whose elements are all $\frac{1}{N}$, where N is the total number of web pages.

i. Interpret the solutions $x \geq 0$ to

$$Bx = x.$$

ii. If you want to avoid multiple solutions, do the same perturbation as in the previous problem. Propose a solution to the obvious problem with billions of web pages to actually computing x and also finding all the information about all the outward hyperlinks.

2. Multivariate calculus

   (a) Show that for the CES -production function $Y(K,L) = A\big(\alpha K^{\rho}+(1-\alpha)L^{\rho}\big)^{\frac{1}{\rho}}$ the limiting marginal rate of technical substitution as $\rho \to 0$ is the Cobb-Douglas production function. In other words, show that:

$$\lim_{\rho \to 0} Y(K,L) = AK^{\alpha}L^{1-\alpha}$$

   (Hint: take logarithms on both sides and use l'Hopital's rule.)

   (b) Find the partial derivatives to

      i.
$$f(x,y,z) = e^{ax-by} - z,$$

      ii.
$$f(x,y) = 6x^{\frac{2}{3}}y^{\frac{1}{2}},$$

      iii.
$$f(x,y,z) = \sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2},$$

   (c) Compute the Jacobian matrix to the vector-valued function for an arbitrary point $x = (x_1, x_2, x_3)$

$$f(x_1,x_2,x_3) == \begin{pmatrix} 2x_1^2x_2^2 - 3x_3 \\ \sqrt{x_1x_2x_3} \end{pmatrix}.$$

3. A firm has a CES -production function

$$Y = \left[\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}\right]^2,$$

   where Y is the output, K the capital input, L the labor input.

   (a) Compute the marginal products for K and L at 36 units of K and 9 units of L.

   (b) The price of output per unit is 9 the cost of a unit of capital is 2 and a labor unit costs 5 euros. The management decides to increase the budget for inputs by Δ, i.e. the production floor may buy additional units of K and L at the above prices . What is the optimal way of using the increased budget (i.e. more capital or more labor or...)? (Hint: use the gradient)