Mathematics for Economists
ECON-C1000
Spring 2020
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Problem Set 2: Due April 29, 2020

1. Chain rule and implicit functions:
   (a) Let $f(x,y) = 6x^2y^2 + xy, x(t) = 10t^2 + 1$ and $y(t) = t^3 + 2t$. Compute the derivative of $f(x(t), y(t)$ with respect to t first by chain rule and then by plugging in the formulas for $x(t), y(t)$ and taking the derivative. Compare your solutions (and the ease of getting at the solutions).
   (b) Consider the equation
   $$f(x,y,w) = y^3x^2 + w^3 + xyw - 3 = 0.$$
   Treat y as the endogenous variable. Can you use implicit function theorem around $(x,y,w) = (1,1,1)$. Compute $\frac{dy}{dx}$ around this point.

2. Consider the system of equations:
$$\frac{\alpha}{y_1} - y_3z_1 = 0$$
$$\frac{\beta}{y_2} - y_3z_2 = 0$$
$$z_1y_1 + z_2y_2 - z_3 = 0$$
   (a) Show that the system is satisfied at point
   $$(y_1, y_2, y_3, z_1, z_2, z_3) = (1, 1, 1, \alpha, \beta, \alpha + \beta)$$
   for $\alpha, \beta > 0$
   (b) Show that you can take (y_1, y_2, y_3) as endogenous variables and use the implicit function theorem there.

3. The profit π for a monopoly firms is computed as follows as a function of the output $q \geq 0$ of the firm:
$$\pi(q) \quad = \quad p(q)\,q - c(q)\,,$$
   where $p(q) = a - bq, a > 0, 0 < b < 1$ is the inverse demand function and $c(q) = \delta q^2$ is the cost function of the firm.
   (a) Compute the optimal level of production for the firm.
   (b) A tax $\tau \in (0,1)$, per unit sold is set. Thenow receives $(1-\tau)\,p$ for each of the units it sells. How does the tax affect the optimal production level?

4. Find the points where the gradient of the following functions is zero. We'll see later in this course how to decide if these points are minima or maxima.

(a) $f(x,y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$

(b) $f(x,y) = x + 2e^y - e^x - e^{2y}$

5. (Implicit function theorem without specified functional forms) A competitive firm maximizes its profit by choosing optimally its inputs:

$$\max_{k,l \geq 0} pf(k,l) - wl - rk,$$

where $f(k,l)$ is the production function, k is capital, l is labor, r is the rental cost of capital, w the market wage for labor and p is the output price.

$$\frac{\partial f(k,l)}{\partial k} = \frac{r}{p}$$
$$\frac{\partial f(k,l)}{\partial l} = \frac{w}{p}$$

To keep the notation simple, denote the marginal products by $\frac{\partial f(k,l)}{\partial k} = g(k,l)$ and $\frac{\partial f(k,l)}{\partial l} = h(k,l)$, and write the conditions for optimality as:

$$g(k,l) - \frac{r}{p} = 0,$$
$$h(k,l) - \frac{w}{p} = 0.$$

(a) What are the natural endogenous variables for this model? What are the exogenous variables?

(b) Assume that the gradient is zero at $\left(\bar{k}, \bar{l}, \bar{p}, \bar{r}, \bar{w}\right)$. When can you use the implicit function theorem to determine the changes in the endogenous variables of small changes in the exogenous variables?

(c) Assume further that for all $k, l \geq 0$, the functions g and h are differentiable and that we have

$$\frac{\partial g(k,l)}{\partial k} < 0, \quad \frac{\partial h(k,l)}{\partial l} < 0$$

and

$$\frac{\partial g(k,l)}{\partial k}\frac{\partial h(k,l)}{\partial l} > \frac{\partial h(k,l)}{\partial k}\frac{\partial g(k,l)}{\partial l}$$

Interpret these assumptions.

(d) Determine the sign of the following

$$\frac{dk}{dr}, \text{ ja } \frac{dl}{dr}$$

By using the implicit function theorem. Does your answer depend on the sign of $\frac{\partial h(k,l)}{\partial k}$? Hint: Use Cramer's rule.