Mathematics for Economists

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Solutions to the problem set 1

Question 1:

a)

i) Demand function: $q^d = a - bp$
Supply function: $q^s = c + dp$

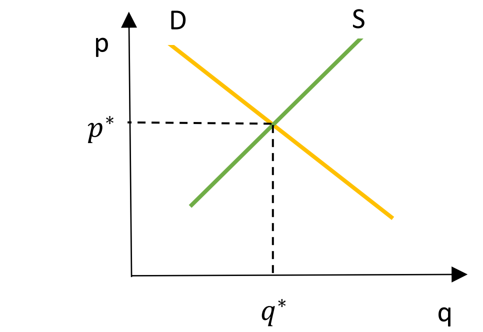
Market clearing condition implies that:

$$q^d = q^s = q$$

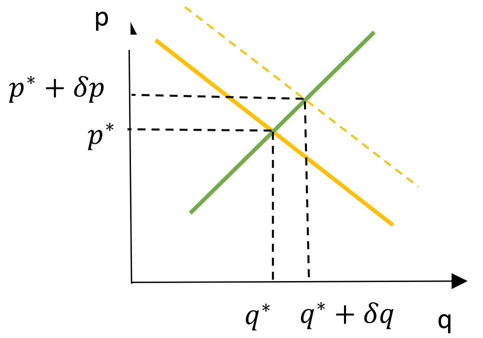
Consequently:

$$q = a - bp = c + dp \Rightarrow a - c = p(b + d) \Rightarrow p^* = \frac{a - c}{b + d}$$

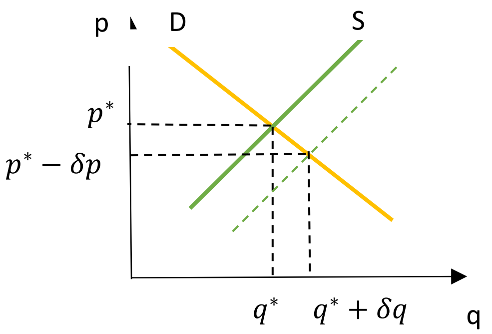
$$q^* = a - b\left(\frac{a - c}{b + d}\right) = \frac{ab + ad - ba + bc}{b + d} = \frac{ad + bc}{b + d}$$



The equilibrium point from the market clearing condition



Increasing the parameter a is equivalent to the upward shift in the demand function. The amount of the shift is equal to δa. Because of this shift, the Equilibrium price and the Equilibrium quantity will be shifted respectively δp and δq.



On the other hand, increasing the parameter c is equivalent to shifting the supply function. As the result, the equilibrium price and the Equilibrium quantity will shift as well.

iv) We first solve the problem for the change in a:

$$a \to a + \delta a \;\Rightarrow p + \delta p = \frac{a + \delta a - c}{b + d} \Rightarrow \; \delta p = \frac{\delta a}{b + d}$$

$$q + \delta q = \frac{(a + \delta a)d + bc}{b + d} \Rightarrow \delta q = \frac{\delta a d}{b + d}$$

And finally:

$$\frac{\delta q}{\delta p} = d$$

Now for the change in c:

$$c \to c + \delta c \;\Rightarrow p + \delta p = \frac{a - c - \delta c}{b + d} \Rightarrow \; \delta p = \frac{-\delta c}{b + d}$$

$$q + \delta q = \frac{ad + b(c + \delta c)}{b + d} \Rightarrow \delta q = \frac{b\delta c}{b + d}$$

So:

$$\frac{\delta q}{\delta p} = -b$$

v) Well we can not say a lot about supply curve if we observe only one pair of price or quantity, but if we have enough data (assume that you observe the market for a period of time), then we will see demand and supply shifters as a result of what happens in the market. For example: consider the current situation with the coronavirus spreading throughout the world, we will definitely see an enormous shift in the demand of the hand washing soap or hand sanitizer. On the other hand, for an example about the shift in the supply curve, just consider opening an export market for a good which will shift the supply curve of that good upward.

b)

i) one way to show that a matrix has a full rank or not is to calculate the determinant of that matrix. If the determinant is equal to 0 then the matrix does not have full rank. Otherwise, the rows and columns of the matrix are linearly independent and it has full rank. So in here:

$$M = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$$

And the determinant of the matrix M is:

$$\det(M) = 0.3 * \det\begin{bmatrix} 0.1 & 0.5 \\ 0.4 & 0.1 \end{bmatrix} - 0.5 * \det\begin{bmatrix} 0.4 & 0.5 \\ 0.3 & 0.1 \end{bmatrix} + 0.4 * \det\begin{bmatrix} 0.4 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}$$

$$\Rightarrow \det(M) = 0.3(0.01 - 0.2) - 0.5(0.04 - 0.15) + 0.4(0.16 - 0.03) = 0.053 \neq 0$$

So the matrix M has full rank. Now for the second matrix N, which is:

$$N = \begin{bmatrix} -0.7 & 0.5 & 0.4 \\ 0.4 & -0.9 & 0.5 \\ 0.3 & 0.4 & -0.9 \end{bmatrix}$$

Although it is possible to calculate the determinant to realize whether the matrix has full rank or not, We can already say that the matrix N does not have full rank, because the rows of this matrix are not linearly independent and:

$$First\ row + Second\ row = -Third\ row$$

ii) Assume that we have a stochastic matrix A

$$A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

So for every column of the matrix we have:

$$\sum_{i=1}^{n} x_{ij} = 1$$

Now we define a matrix B, which is equal to $A - I$.

$$B = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}$$

Now because B=A-I we can easily conclude that for every column of the matrix B:

$$\sum_{i=1}^{n} y_{ij} = 0$$

And this means that the rows of the matrix B are not linearly independent from each other and it is easy to see that for any k:

$$[x_{k1} \quad \cdots \quad x_{kn}] = -\sum_{i \neq k} [x_{i1} \quad \cdots \quad x_{in}]$$

So the matrix B does not have full rank.
We can get the exact same results for the matrix $C = I - A$.

c)
i- We set

$$A_j = \sum_{i=1}^{30} a_{ji}$$

It is obvious that A_j is a real number and it shows the total amount of the endorsement that student j gets from his classmates.
ii- Now we set

$$B_i = \sum_{i=1}^{30} a_{ij}$$

According to the question, B_i can have two possible values. If $B_i = 0$ it means that student i did not propose any endorsements for his classmates. On the other hand, if it is equal to one it means that student i made some endorsements for his classmates. Note that in this part endorsements from different students have the same weights.
iii- It is easy to see that the matrix A is a stochastic matrix, and the sum of the elements of each column is equal to one (Note that we assumed here that all of the students should give endorsements about their classmates). So as we have seen before if A is a stochastic matrix then $A - I$ does not have full ranks.
iv- Consider the case where we divide the class of 30 students into 15 groups and each group consists of only two students and these two only endorse each other and no one

else. For example we can assume that student 1 only endorses student 2 and vice versa, and we have the same situation for students 3 and 4 till we get to students 29 and 30. Reminding the main equation:

$$Ax = x$$

We immediately realize that with the above conditions we will get the following results:

$$x_1 = x_2, x_3 = x_4, \dots \ , x_{29} = x_{30}$$

Moreover, we have the condition that $\sum_{i=1}^{30} x_i = 30$.

Consequently, we have 30 variables but we only have 16 equations to solve them, and this means that we have infinite solutions for these equations.

d)

i) Same as before, we can consider x to be the weight of the different websites. One concept of weight for the websites can be the number of the times that people visit that specific website in a time, which will increase the number of the time that people click on the links on that specific website.

ii)

As one of the results of the Perron-Frobenius Theorem:

Let B be a positive matrix (the reason that we perturb the matrix B to get B'), then B' has exactly one positive eigenvector x^0. If λ_0 is its eigenvalue, then $\lambda_0 > 0$ and if B' is stochastic $\lambda_0 = 1$.

Now we know that B is a matrix that mentions which websites are connected to each other in the way that they include each other's links, but when we perturb B to get B' we get a probability distribution matrix. In other words, what we can say now is that websites i includes the link of the website j with probability B'_{ij} which is positive and between zero and one.

According to another result of the Perron-Frobenius theorem for the Markov chains, If we start with any arbitrary nonzero ranking vector x', we will get to x^0 after enough transitions. In other words:

$$x^0 = \lim_{i \to \infty} B'^i x'$$

You will see more of these dynamical systems in the course.

It is necessary to mention that Google use the same strategy to rank the webpages. To better understand this consider one webpage with index i. Google go through this webpage and extract every link that it contains (using crawlers). If we assume that the total number of the links in this webpage is N_i, Google set P_{ij} with $\frac{1}{N_i}$ for every time he finds the link to the webpage j (measure the frequency of the webpage j). To get the ranking of the webpage i, Google use the same formulation as before:

$$x_i = \sum_j P_{ij} x_j$$

Where $x_i \ and \ x_j$ are the rankings of the webpages i and j respectively, and Google does this for millions and billions of the webpages. Using Perron-Frobenius and using the fact that matrix P is positive (using the perturbation) there is a unique solution to this problem.

Question 2:

a) $Y(K, L) = A(\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}}$

$$\lim_{\rho\to 0} Y(K,L) = ?$$

To solve this problem we first take logarithms from both sides and then use l'Hopital's rule. So

$$\log Y = \log A + \frac{\log(\alpha K^\rho + (1-\alpha)L^\rho)}{\rho}$$

$$\lim_{\rho\to 0} \log Y(K,L) = \log A + \lim_{\rho\to 0} \frac{\log(\alpha K^\rho + (1-\alpha)L^\rho)}{\rho}$$

using l'Hopital's rule, we have

$$\lim_{\rho\to 0} \log Y(K,L) = \log A + \lim_{\rho\to 0} \frac{(\alpha K^\rho \log K + (1-\alpha)L^\rho \log L)}{\alpha K^\rho + (1-\alpha)L^\rho}$$

$$= \log A + \alpha \log K + (1-\alpha)\log L = \log(AK^\alpha L^{1-\alpha})$$

$$\Rightarrow \lim_{\rho\to 0} Y(K,L) = AK^\alpha L^{1-\alpha}$$

b)

i) $f(x,y,z) = e^{ax-by} - z$

The partial derivatives are:

$$f_x = a.e^{ax-by}$$
$$f_y = -b.e^{ax-by}$$
$$f_z = -1$$

ii) $f(x,y) = 6x^{\frac{2}{3}}y^{\frac{1}{2}}$

The partial derivatives are:

$$f_x = 4x^{-\frac{1}{3}}y^{\frac{1}{2}}$$
$$f_y = 3x^{\frac{2}{3}}y^{-\frac{1}{2}}$$

iii) $f(x,y,z) = \sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}$

The partial derivatives are:

$$f_x = \frac{\frac{1}{4}x^{-\frac{1}{2}}}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}$$

$$f_y = \frac{\frac{1}{6}y^{-\frac{2}{3}}}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}$$

$$f_z = \frac{5z}{\sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2}}$$

c)

Jacobian of a matrix f is defined as follows:

$$J = \left[\frac{\partial f}{\partial x_1}, \quad \dots, \quad \frac{\partial f}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

And for the function

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 2x_1^2 x_2^2 - 3x_3 \\ \sqrt{x_1 x_2 x_3} \end{pmatrix}$$

The Jacobian is equal to:

$$J(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 4x_1 x_2^2 & 4x_1^2 x_2 & -3 \\ \frac{\frac{1}{2} x_2 x_3}{\sqrt{x_1 x_2 x_3}} & \frac{\frac{1}{2} x_1 x_3}{\sqrt{x_1 x_2 x_3}} & \frac{\frac{1}{2} x_1 x_2}{\sqrt{x_1 x_2 x_3}} \end{pmatrix}$$

Question 3:

a) $Y = (\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L})^2$

$$Marginal\ products\ for\ K = \frac{\partial Y}{\partial K} = 2 * \frac{2}{3} * \frac{1}{2} * \frac{1}{\sqrt{K}} * \left(\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}\right) =$$

$$= \frac{2}{3} * \frac{\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}}{\sqrt{K}} = \frac{5}{9}$$

$$Marginal\ products\ for\ L = \frac{\partial Y}{\partial L} = 2 * \frac{1}{3} * \frac{1}{2} * \frac{1}{\sqrt{L}} * \left(\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}\right) =$$

$$= \frac{1}{3} * \frac{\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}}{\sqrt{L}} = \frac{5}{9}$$

b) If we want to increase the budget of a firm, first we should pay attention to the marginal products of the inputs (so which input will increase the output more) and second to the prices of the inputs. Consequently, we derive the marginal product divided by input price for two inputs and the one which has the higher values will be chosen to be increased. So: for Capital:

$$\frac{\frac{1}{6} * \frac{\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}}{\sqrt{K}}}{P_k} = \frac{\frac{5}{9}}{2} = \frac{5}{18}$$

And for Labor:

$$\frac{\frac{1}{3} * \frac{\frac{2}{3}\sqrt{K} + \frac{1}{3}\sqrt{L}}{\sqrt{L}}}{p_L} = \frac{\frac{5}{9}}{5} = \frac{1}{9}$$

So Capital has higher marginal values divided by the price, so we will increase the Capital if we want to invest Δ in the firm.