Mathematics for Economists
ECON-C1000
Spring 2020
Juuso Välimäki
juuso.valimaki@aalto.fi

Problem Set 3: Due May 6, 2020

1. Consider the following system of equations:

$$\begin{aligned} f_1(y_1, y_2; \alpha, \beta, q) &= y_1 + \alpha y_2^2 - q = 0 \\ f_2(y_1, y_2; \alpha, \beta, q) &= \beta y_1^2 + y_2 - q = 0 \end{aligned}$$

   (a) Can you solve (y_1, y_2) as functions of (α, β, q) in a neighborhood of the point $(y_1, y_2) = (1, 1)$, $(\alpha, \beta, q) = (1, 1, 2)$?

   (b) Find the matrix of partial derivatives:

$$\begin{pmatrix} \frac{\partial y_1}{\partial \alpha} & \frac{\partial y_1}{\partial \beta} & \frac{\partial y_1}{\partial q} \\ \frac{\partial y_2}{\partial \alpha} & \frac{\partial y_2}{\partial \beta} & \frac{\partial y_2}{\partial q} \end{pmatrix}.$$

2. A monopolist firm sells two models of a smartphone. Buyers prefer the fancier mode, but if the price is too high, they switch to the cheaper model. Let (p_1, p_2) denote the prices of the fancy and cheap phone respectively with $p_1 > p_2 \geq 0$. The demand at a given pair of prices is $q_1(p_1, p_2, x_1), q_2(p_1, p_2, x_2)$. The costs of production for the two models are given by: $c_1 q_1^2, c_2 q_2^2$.

   (a) The monopoly firm chooses the two prices to maximize total sales revenue net of the production cost. Write the objective function of the monopoly firm.

   (b) What are the natural assumptions to make on the partial derivatives of the demand functions?

   (c) Write the first order necessary condition for the optimal pair of prices $(\hat{p}_1, \hat{p}_2)$

   (d) Suppose that $q_1 = x_1 + \alpha_1 p_1 + \beta_1 p_2$, $q_1 = x_2 + \alpha_2 p_1 + \beta_2 p_2$ and solve the optimal prices explicitly. What do you need to assume about the α_i, β_i, x_i parameters for the model to make sense?

3. Consider the following matrices and

   (a) Determine if the following matrices are positive or negative definite:

$$\text{i)} \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} -1 & -2 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & -9 \end{pmatrix}$$

(b) For which values of b is the matrix $A = \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$ positive definite?

(c) Minimize the function

$$f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2)$$

for some value of b that you found in the previous part. Is the minimum global minimum?

4. Which of the following functions is concave?

   (a) $f(x) = x^3$.

   (b) $f(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

   (c) $f(x, y) = -x^2 - y^2 + 3xy$.

5. Consider the general **Cobb-Douglas** function defined for all $x_1, x_2 > 0$ by

$$f(x_1, ..., x_n) = Ax_1^{\alpha_1} x_2^{\alpha_2}$$

where $A, \alpha_1, \alpha_2,$ are positive constants. Find sufficient conditions for f to be

   (a) concave and strictly concave,

   (b) quasiconcave and strictly quasiconcave.