Mathematics for Economists: Lecture 6

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This lecture covers

1. Positive and negative definite quadratic forms
2. Concave and convex functions
3. Convex sets
4. Quasiconcave functions

Quadratic forms and the definiteness of matrices

- A quadratic form is a homogenous second-degree polynomial whose terms are all of second order. They can be written as:

  $$x \cdot Ax$$

  for some symmetric matrix A.
- A quadratic form is *positive definite* if for all $x \neq 0$, $x \cdot Ax > 0$. It is *positive semidefinite* if for all x, $x \cdot Ax \geq 0$.
- A quadratic form is *negative definite* if for all $x \neq 0$, $x \cdot Ax < 0$. It is *negative semidefinite* if for all x, $x \cdot Ax \leq 0$. In all other cases, we say that the quadratic form is indefinite.
- By multivariate Taylor's theorem, definiteness of Hessian matrix is useful in classifying critical points of multivariate functions.

Quadratic forms and the definiteness of matrices

- Let's consider bivariate quadratic functions $f(x_1, x_2)$ with matrix A:

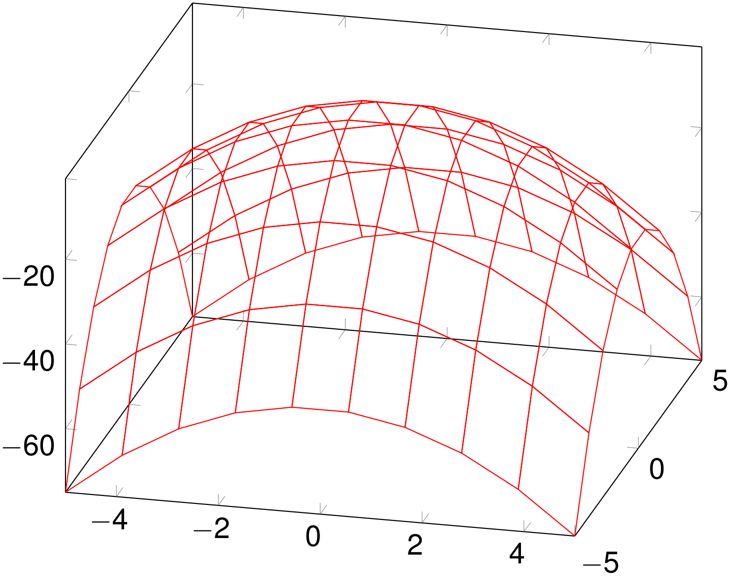
$$f(x_1, x_2) = (x_1, x_2)A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ with } A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

- So the quadratic form is:

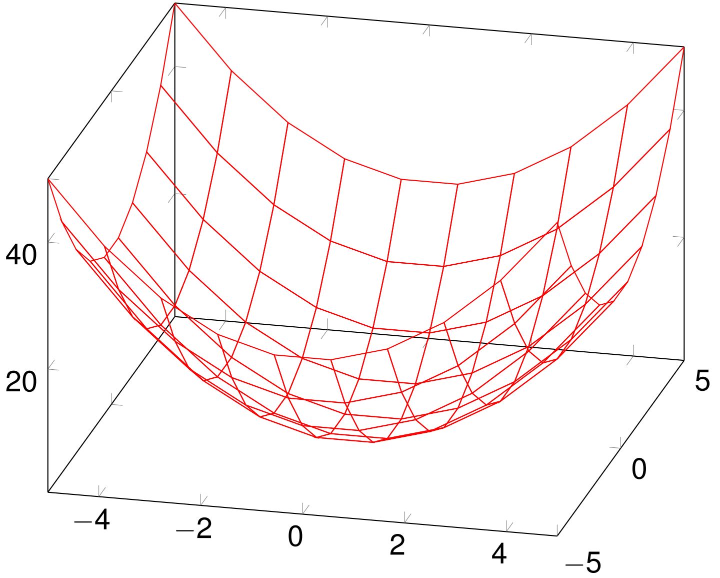
$$ax_1^2 + 2bx_1x_2 + cx_2^2.$$

- What do these functions look like?
- All have critical point at $(0, 0)$

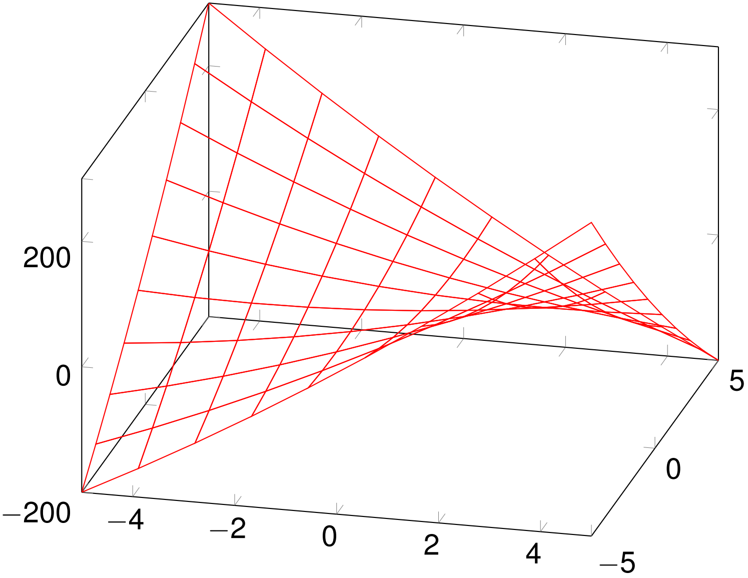
$f(x_1, x_2) = -x_2^2 - 2x_1^2$: maximum at $(0, 0)$



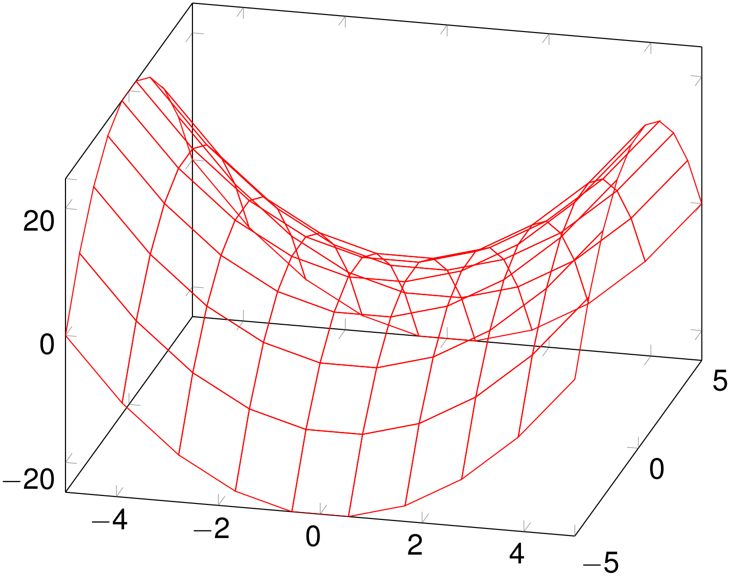
$f(x_1, x_2) = x_2^2 + x_1^2$: minimum at $(0, 0)$



$f(x_1, x_2) = x_2^2 + x_1^2 - 10x_1x_2$: saddle at $(0, 0)$



$f(x_1, x_2) = x_2^2 - x_1^2$: saddle at $(0, 0)$



Quadratic forms and the definiteness of matrices

- When do we have for all $(x_1, x_2) \neq (0, 0)$
$$ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$$
or
$$ax_1^2 + 2bx_1x_2 + cx_2^2 < 0$$
- We see immediately that when $b = 0$, the quadratic form is a sum of squares. Hence the classification depends on the signs of a, c. If They are both positive, we have positive definite matrix, if both negative, we have negative definite matrix. If different signs, indefinite matrix.

Quadratic forms and the definiteness of matrices

- View $ax_1^2 + 2bx_1x_2 + cx_2^2$ as a function of x_2. If $c < 0$, then the function has a minimum at $x_2 = -\frac{bx_1}{c}$.
- Plugging into the quadratic form:

$$ax_1^2 - 2\frac{b^2x_1^2}{c} + \frac{b^2x_1^2}{c} = \left(a - \frac{b^2}{c}\right)x_1^2.$$

- This is strictly positive for all $x_1 \neq 0$ if

$$\begin{aligned} \left(a - \frac{b^2}{c}\right) &> 0 \text{ or} \\ ac &> b^2. \end{aligned}$$

- In other words, the quadratic form is positive definite if i) $a, c > 0$ and ii) $\det A > 0$. For semidefiniteness, the inequalities are weak.

Quadratic forms and the definiteness of matrices

- For negative definiteness, assume that $a, c < 0$. Solving for the maximal x_2 for eachx_1 gives:

$$x_2 = -\frac{bx_1}{c}$$

  and plugging into the quadratic form and require that:

$$ax_1^2 - 2\frac{b^2x_1^2}{c} + \frac{b^2x_1^2}{c} = \left(a - \frac{b^2}{c}\right)x_1^2 < 0.$$

- We get:

$$a < \frac{b^2}{c} \text{ or } ac > b^2.$$

- In other words, we need that $a, c < 0$ and $\det A > 0$
- For semidefiniteness, the inequalities in the above are weak inequalities.

Observations on the general case

- Take an arbitrary symmetric $n \times n$ matrix A. If $a_{ii} > 0$ and $a_{jj} < 0$ for some $i, j \in \{1, ..., n\}$, then the matrix is indefinite. To see this, just compute

$$\mathbf{e}_i \cdot A\mathbf{e}_i = a_{ii}, \qquad \mathbf{e}_j \cdot A\mathbf{e}_j = a_{jj}.$$

- The general method for determining the definiteness based on determinants of some submatrices of A is given in the notes.
- Later in the course, we will talk about eigenvalues of matrices and will see that positive definiteness of a matrix is equivalent to strictly positive eigenvalues and negative definiteness to strictly negative eigenvalues. (For semidefiniteness, the eigenvalues must be non-negative and non-positive respectively).

Convex and concave functions: Convex set

- We start with a definition

Definition

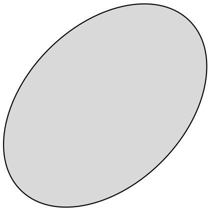
A set X *is convex* if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, we have:

$$\lambda x + (1 - \lambda) y \in X.$$

We call $\lambda x + (1 - \lambda) y$ a *convex combination* of x and y.

- On the real line, convex sets are intervals $a \leq x \leq b$ for some $-\infty \leq a \leq b \leq \infty$.
- In $\mathbb{R}^n$, convex sets are sets X with the property that when you connect linearly two points in X, the entire connecting line is also in X.
- Hence a disk in the plane is convex and a cube in the three dimensional space are convex, but the circle in the plane is not, a disk with the center removed is not, a doughnut in three dimensions is not etc.

[Convex set]



[Non-convex set]

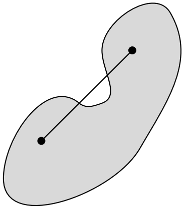


Figure: Graphical interpretation of convex sets.

Convex and concave functions: Definitions

- Consider a real-valued function $f : X \to \mathbb{R}$, where X is a convex set.

Definition

The function f *is convex* if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, we have:

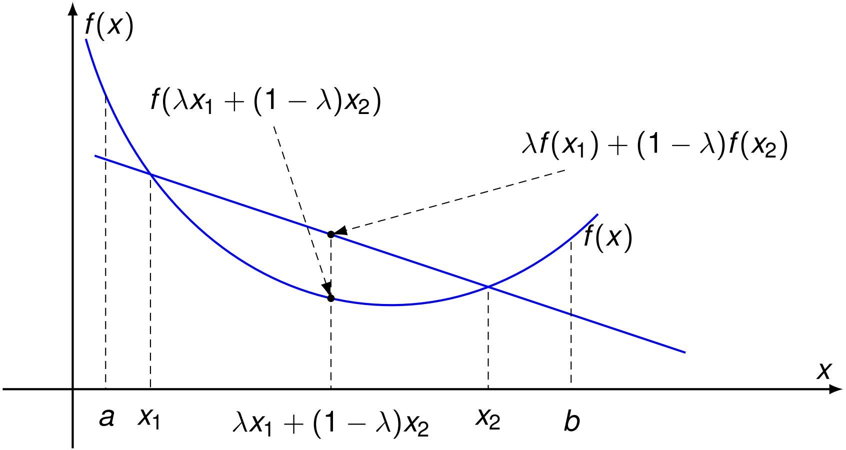
$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

f *is concave* if

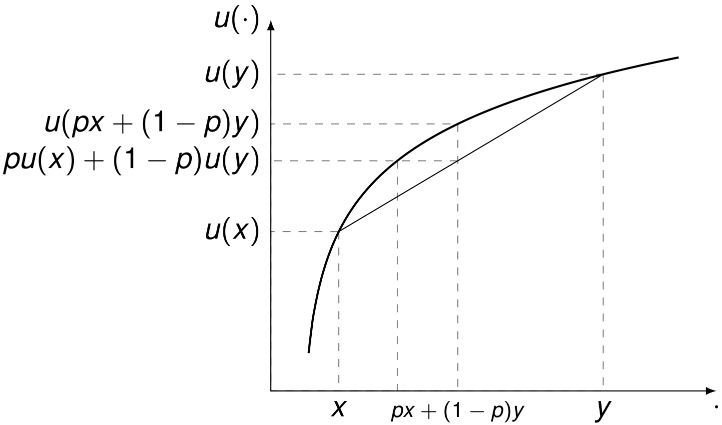
$$f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y).$$

- Note: If f is convex, then $-f$ is concave

A convex function



A concave utility function



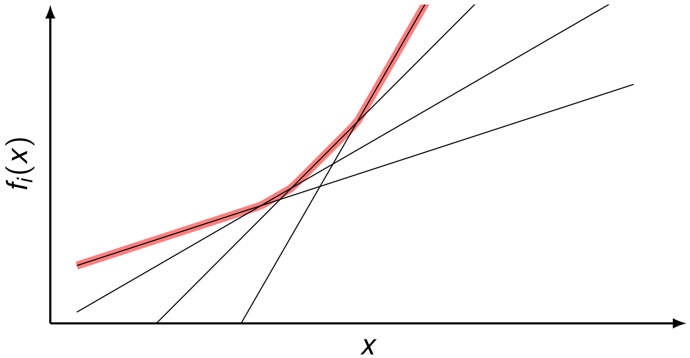
Properties of convex functions

- If $f(x)$ is convex, then $g(x) = -f(x)$ is concave.
- If $f(x)$ is convex, then $af(x)$ is convex if $a > 0$.
- If $f(x)$ and $g(x)$ are convex, then $h(x) = f(x) + g(x)$ is convex.
- If $f(x)$ and $g(x)$ are convex, then $h(x) = f(x)\,g(x)$ is not necessarily convex. (Give an example for both cases, i.e. where the product of convex functions is convex and where it is not).
- Exercise: Assume that $f : X \to \mathbb{R}$ is convex and $g : \mathbb{R} \to \mathbb{R}$ is also convex. Is $g(f(x))$ convex? What if g is increasing and convex?
- (Optional Exercise): Assume that $f : X \to \mathbb{R}$ is a convex function. Show that the set

  $$\{(x, y) \in \mathbb{R}^{n+1} \quad | \quad x \in X, y \geq f(x)\}$$

  is a convex set.

Maximum of linear functions is convex



Properties of convex functions

One of the most important results is the following:

Proposition

If $f(x)$ and $g(x)$ are convex, then $h(x) = \max\{f(x), g(x)\}$ is convex.

Proof: Since by assumption, f and g are convex, we have:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

and

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y).$$

Properties of convex functions

By definition,

$$
\begin{aligned}
h(\lambda x+(1-\lambda)y) &= \max\{f(\lambda x+(1-\lambda)y), g(\lambda x+(1-\lambda)y)\} \\
&\leq \max\{\lambda f(x)+(1-\lambda)f(y), \lambda g(x)+(1-\lambda)g(y)\} \\
&\leq \lambda\max\{f(x), g(x)\}+(1-\lambda)\max\{f(y), g(y)\} \\
&= \lambda h(x)+(1-\lambda)h(y).
\end{aligned}
$$

The first inequality follows from the convexity of f and g.The second follows by choosing the larger of $f\cdot, g(\cdot)$ for x, y. The last equality is the definition of h.

Properties of convex functions

- The same result is true for an arbitrary set of convex functions. Let $f(x;\alpha)$ be convex in $x$for all α. Then
$$g(x) = \max_{\alpha} f(x;\alpha)$$
is convex. The proof is identical to the one above.
- Since linear functions are convex, this result holds for any set of linear functions.
- Since
$$\max\{f(x), g(x)\} = -\min\{-f(x), -g(x)\},$$
and since $-f$ is concave when f is convex, we get:
$$g(x) = \min_{\alpha} f(x;\alpha)$$
is concave if $f(x;\alpha)$ is concave in x for all α.

Economic examples

- **Profit function of a firm**
- A competitive firm sells output y at price p_0 and buys inputs $x = (x_1, ..., x_n)$ at input prices $(p_1, ..., p_n)$. Its profit is

$$p_0 y - \Sigma_{i=1}^{n} p_i x_i.$$

- The maximization problem is then

$$\max_{y,x \in F} p_0 y - \Sigma_{i=1}^{n} p_i x_i,$$

  where F is the feasible set determined by technological possibilities.
- The profit function of the firm gives the maximum achievable profit amongst the feasible set at input and output prices p.

$$\pi(p) = \pi(p_0, p_1, ..., p_n) = \max_{y,x \in F} p_0 y - \Sigma_{i=1}^{n} p_i x_i$$

- Since the profit from a fixed feasible production is a linear function of the prices p, the profit function is the maximum over linear functions and therefore convex in p.

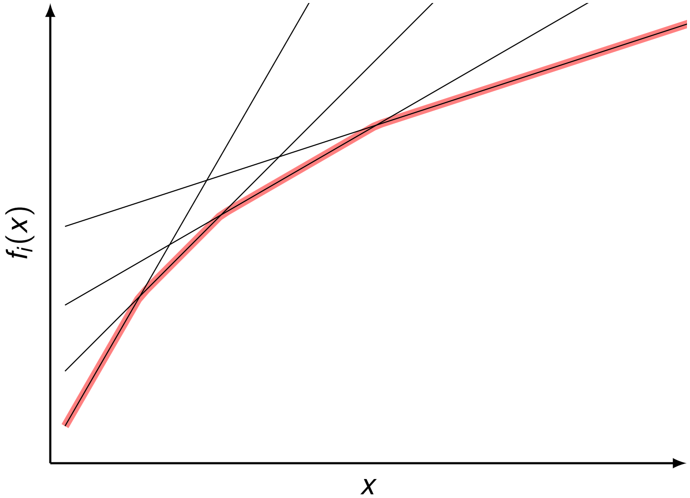
Economic examples

- **Expenditure minimization**
- Let X be the feasible set for inputs $x = (x_1, ..., x_n)$ and $p = (p_1, ..., p_n)$ be the input prices.
- The expenditure function

$$e(p; X) = \min_{x \in X} p.x = \min_{x \in X} \Sigma_{i=1}^{n} p_i x_i$$

is a concave function by the same argument as above.
- These two examples show that convexity and concavity play a real role in economic applications.
- We shall see more applications when we discuss constrained optimization and value functions of optimization problems.
- Is there an economic intuition for the maximum of linear functions being convex? We'll return to this after some further characterizations of convex functions.

Lower envelope of linear functions is concave



Convexity and concavity of differentiable functions

- When $f : \mathbb{R} \to \mathbb{R}$, and f is convex and differentable, it it is easy to see by drawing a picture that for all x, y we have:

$$f(y) - f(x) \geq f'(x)(y - x).$$

- This just says that the graph $(x, f(x))$ of a convex function f is above all of its tangent lines.
- Similarly, the graph of a concave function lies below its tangent line

Convexity and concavity of differentiable functions

The multi-dimensional version is proved in the notes:

Proposition

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ *is convex if and only if*

$$f(y) - f(x) \geq Df(x)(y - x) \quad \textit{for all } x, y.$$

Can you formulate this condition in terms of level curves and gradients? What is the corresponding result to concave functions?

Second derivatives and convexity

- Start again with functions of a single variable. By Taylor's theorem without the remainder term,

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2}f''(x)(y - x)^2 + \frac{1}{6}f'''(x)(y - x)^3 + \ldots$$

- In order to have

$$f(y) - f(x) \geq f'(x)(y - x)$$

  for $|y - x|$ small, we must have

$$f''(x) \geq 0.$$

- In other words, convex functions have a positive second derivative.

Second derivatives and convexity

- Taylor's theorem with a remainder term of second degree:

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2} f''(z)(y - x)^2$$

  for some $z \in [x, y]$.
- If f'' is everywhere non-negative, we get:

$$f(y) - f(x) \geq f'(x)(y - x)$$

  for all y, x and f is therefore convex.
- Let's generalize now to $f : X \to \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$ n.
- Convexity corresponds to positive semidefiniteness of the Hessian matrix.
- Concavity corresponds to negative semidefiniteness of the Hessian matrix.
- Hence we see an immediate connection between convexity and the second order conditions for optimality.

Quasiconvex and quasiconcave functions

- Even though the name suggests something extremely technical and tedious, quasiconcavity is actually one of the most important notions for functions in economic theory.

Definition

- A function f on a convex set X is *quasiconcave* if for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda) y) \geq \min\{f(x), f(y)\}.$$

f is *quasiconvex* is for all $x, y \in X$ and for all $\lambda \in [0, 1]$

$$f(\lambda x + (1 - \lambda) y) \leq \max\{f(x), f(y)\}.$$

Exercise: f is quasiconcave, then $-f$ is quasiconvex.

Quasiconvex and quasiconcave functions: Observations

- If f is quasiconcave, then af is quasiconcave if $a > 0$.
- If f and g are quasiconcave $f + g$ is not necessarily quasiconcave.
- All monotone (i.e. all increasing and all decreasing) functions of a single variable are both quasiconcave and quasiconvex. This is NOT true for multidimensional functions
- All concave functions are quasiconcave. Show this as an exercise.
- Not all quasiconcave functions are concave.
- If f is a quasiconcave function and g is a strictly increasing function, then $h(x) = g(f(x))$ is a quasiconcave function.

Quasiconvex and quasiconcave functions: Observations

- An upper contour set of function f for value α is denoted by $U(f;\alpha)$ and defined as:
$$U(f;\alpha) := \{x \in X | f(x) \geq \alpha\}.$$
- Interpretation: if f is a utility function, $U(f;\alpha)$ is the better side of the indifference curve giving utility level α.

Proposition

A function f is quasiconcave if and only if $U(f;\alpha)$ is a convex set for all α.

Quasiconvex and quasiconcave functions: Observations

Definition

A function f on a convex set X is *strictly quasiconcave* if for all $x, y \in X$ and for all $\lambda \in (0,1)$

$$f(\lambda x + (1-\lambda) y) > \min\{f(x), f(y)\}.$$

- The following exercise shows why strict quasiconcavity is very useful for optimization problems.
- Exercise: show that if a strictly quasiconcave function has a maximum, then the maximum is unique.

Quasiconcavity and differentiability

- A differentiable function f on a convex set X is quasiconcave if and only if:

$$f(y) \geq f(x) \Rightarrow Df(x)(y - x) \geq 0.$$

- Exercise: Compare this to the definition of concavity for differentiable functions and relate this condition to the geometry of upper contour sets and tangent planes to the upper contour sets.