Mathematics for Economists

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Solutions to the problem set 2:

Question 1:

a) First we try to obtain the derivative using the chain rule:

$$\frac{d}{dt}f(x,y) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t)$$
$$= \left(12x(t)y^2(t) + y(t)\right) * (20t) + \left(12x^2(t)y(t) + x(t)\right) * (3t^2 + 2)$$
$$= \left(12(10t^2 + 1)(t^3 + 2t)^2 + (t^3 + 2t)\right) * (20t)$$
$$+ (12(10t^2 + 1)^2(t^3 + 2t) + 10t^2 + 1) * (3t^2 + 2)$$
$$= \left(1 + 12(t^3 + 2t)(10t^2 + 1)\right)(t^3 + 2t)(20t) +$$
$$\left(1 + 12(t^3 + 2t)(10t^2 + 1)\right)(10t^2 + 1)(3t^2 + 2) =$$
$$\left(1 + 12(t^3 + 2t)(10t^2 + 1)\right)[50t^4 + 63t^2 + 2] =$$
$$[1 + 120t^5 + 252t^3 + 24t + 1][50t^4 + 63t^2 + 2]$$

Now by plugging x(t) and y(t) to f(x,y) and taking the derivative:

$$f\left(x(t), y(t)\right) = xy(1 + 6xy) = (10t^2 + 1)(t^3 + 2t)[1 + 6(10t^2 + 1)(t^3 + 2t)]$$
$$= (10t^5 + 21t^3 + 2t)[1 + 60t^5 + 126t^3 + 12t]$$
$$\Rightarrow f'(t) = (50t^4 + 63t^2 + 2)[1 + 60t^5 + 126t^3 + 12t] +$$
$$(10t^5 + 21t^3 + 2t)[300t^4 + 378t^2 + 12] =$$
$$(50t^4 + 63t^2 + 2)([1 + 120t^5 + 252t^3 + 24t + 1])$$

b) Since y is the endogenous function, we can write:

$$f(x, y(x,w), w) = y^3(x,w)x^2 + w^3 + xy(x,w)w - 3 = 0$$

To use implicit function theorem, we need two conditions to be satisfied:

- Function f should be continuously differentiable at the point (1,1,1), which is clearly satisfied
- And $\frac{\partial f(\hat{x},\hat{y},\hat{w})}{\partial y} \neq 0$. To see if we have this condition:
  $\frac{\partial f(x,y,w)}{\partial y} = 3y^2x^2 + xw$ which is equal to 4 at $(1,1,1)$ so we can use the implicit function theorem here. Taking a derivative from function f to x we have:

$$f'(x,w) = \frac{\partial f}{\partial x} + {y'}_x\frac{\partial f}{\partial y} + {w'}_x\frac{\partial f}{\partial w}$$

But we know that the derivative of w with respect to x is equal to zero, so:

$$f'(x,w) = \frac{\partial f}{\partial x} + {y'}_x\frac{\partial f}{\partial y} = 0 \Rightarrow {y'}_x = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3}{4}$$

Question 2:

a) We have the system of equations:

$$\frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$\frac{\beta}{y_2} - y_3 z_2 = 0$$
$$z_1 y_1 + z_2 y_2 - z_3 = 0$$

Now at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1,\alpha,\beta,\alpha+\beta)$$

We have:

$$\alpha - \alpha = 0$$
$$\beta - \beta = 0$$
$$\alpha + \beta - (\alpha + \beta) = 0$$

So the system is satisfied at this point.

b) Lets write the system of the equations in the following form:

$$f_1(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$f_2(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\beta}{y_2} - y_3 z_2 = 0$$
$$f_3(y_1, y_2, y_3, z_1, z_2, z_3) = z_1 y_1 + z_2 y_2 - z_3 = 0$$

And we know that the equations are satisfied at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1,\alpha,\beta,\alpha+\beta)$$

So we make the matrices of the partial derivatives:

$$D_y f(\hat{y},\widehat{z}) = \begin{bmatrix} \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_3} \\ \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_3} \\ \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_3} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{\hat{y}_1^2} & 0 & -\hat{z}_1 \\ 0 & -\frac{\beta}{\hat{y}_2^2} & -\hat{z}_2 \\ \hat{z}_1 & \hat{z}_2 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & -\alpha \\ 0 & -\beta & -\beta \\ \alpha & \beta & 0 \end{bmatrix}$$

$$\det\left(D_y f(\hat{y},\widehat{z})\right) = -\alpha\beta(\alpha+\beta)$$

And since

$$\alpha, \beta > 0$$

$$\det\left(D_y f(\hat{y},\widehat{z})\right) \neq 0$$

So we have the necessary condition to use the implicit function theorem to obtain $\frac{dy}{dz}$.

Now we have:

$$dy = [D_y f(\hat{y},\widehat{z})]^{-1}[D_z f(\hat{y},\widehat{z})]dz$$

Question 3:

$$\pi(q) = p(q)q - c(q)$$

Where p(q) is the inverse demand function and is equal to:

$$p(q) = a - bq$$

And c(q) is the cost function of the firm and it is equal to:

$$c(q) = \delta q^2$$

a)

$$\pi(q) = p(q)q - c(q) = q(a - bq) - \delta q^2$$
$$\frac{d\pi}{dq} = a - 2bq - 2\delta q = 0 \Rightarrow q^* = \frac{a}{2(b + \delta)}$$

b) Now we have the per unit tax, so the profit function is:

$$\pi(q) = (1 - \tau)p(q)q - c(q) = (1 - \tau)q(a - bq) - \delta q^2$$
$$\frac{d\pi}{dq} = a - 2bq - 2\delta q = 0 \Rightarrow q^* = \frac{a(1 - \tau)}{2(\delta + b(1 - \tau))}$$

As you see, introducing tax will decrease the numerator by the factor of $(1 - \tau)$ and decrease the denominator by the factor less than $(1 - \tau)$, so over all the optimal production level will decrease as the result of the tax implementation.

Question 4:

a. $f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 24x^2 + 2y - 6x \\ 2x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So we have a system of two equations. From the second one:

$$x = -y$$

Putting it into the first one we get:

$$24x^2 - 8x = 8x(3x - 1) = 0 \Rightarrow x_1 = 0, x_2 = \frac{1}{3}$$

So the pairs are:

$$(x_1, y_1) = (0,0) \ \ and \ \ (x_2, y_2) = (\frac{1}{3}, -\frac{1}{3})$$

b. $f(x, y) = x + 2e^y - e^x - e^{2y}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 - e^x \\ 2e^y - 2e^{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving the system of the equations, we have just one pair, which is:

$$(x_0, y_0) = (0,0)$$

Question 5:

a) We assume that prices are given from the outside of the firm, so p, w and r are exogenous variables. On the other hand, it is possible for us to calculate the exact amount of the capital and labour to increase out profit so l and k are endogenous variables.

b) We rewrite the problem as follows:

$$f_1(k,l;p,r,w) = g(k,l) - \frac{r}{p}$$
$$f_2(k,l;p,r,w) = h(k,l) - \frac{w}{p}$$

And we know that the system of the equations are satisfied at $(\bar{k},\bar{l},\bar{p},\bar{r},\bar{w})$, so in the next step we make the matrices of partial derivative (we assume y as the endogenous and x as the exogenous variables):

$$D_y f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial l} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial l} \end{bmatrix} = \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}$$

And

$$D_x f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial p} & \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix}$$

Now assuming the fact that functions f and g are continuously differentiable at $(\bar{k},\bar{l},\bar{p},\bar{r},\bar{w})$, the necessary condition is:

$$\det\left(D_y f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w})\right) \neq 0$$

so:

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k} . \frac{\partial h(\bar{k},\bar{l})}{\partial l} \neq \frac{\partial g(\bar{k},\bar{l})}{\partial l} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

c) As we know

$$g(k,l) = \frac{\partial f(k,l)}{\partial k}$$

So

$$\frac{\partial g(k,l)}{\partial k} = \frac{\partial^2 f(k,l)}{\partial^2 k} < 0$$
$$\frac{\partial h(k,l)}{\partial l} = \frac{\partial^2 f(k,l)}{\partial^2 l} < 0$$

And these assumptions mean that we have decreasing marginal products in k and l.

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k} . \frac{\partial h(\bar{k},\bar{l})}{\partial l} > \frac{\partial g(\bar{k},\bar{l})}{\partial l} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

And this assumption simply means that the cross effects of k and l are not too strong over each other. In other words, adding Labor does not change the marginal product of k (MP_k) too much and vice versa.

d) Using implicit function theorem we have:

$$D_y f(\hat{y},\hat{x})dy + D_x f(\hat{y},\hat{x})dx = 0$$

So

$$\begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix} \begin{bmatrix} dk \\ dl \end{bmatrix} + \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix} \begin{bmatrix} dp \\ dr \\ dw \end{bmatrix} = 0$$

Using Cramer's rule:

$$dk = \frac{det \begin{bmatrix} \frac{1}{\rho} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ 0 & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix} dr}{\det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}}$$

From the previous part we know that the denominator of the above fraction is positive so the sign of $\frac{dk}{dr}$ is the same as the sign of

$$\frac{1}{\rho}\frac{\partial h(\bar{k},\bar{l})}{\partial l}$$

We can compute $\frac{dl}{dr}$ by doing the same:

$$dl = \frac{det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{1}{\rho} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & 0 \end{bmatrix} dr}{\det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}}$$

And the sign of $\frac{dl}{dr}$ is the same as the sign of

$$-\frac{1}{\rho}\frac{\partial h(\bar{k},\bar{l})}{\partial k}$$