Mathematics for Economists
ECON-C1000
Spring 2020
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Problem Set 4: Due May 13, 2020

1. Maximize the function $f(x, y) = xy$

   subject to

$$\begin{aligned} x + y &\leq 100 \\ x &\leq 40 \\ x, y &\geq 0 \end{aligned}$$

2. Show that the budget set

   $B(p, w) := \{x \in \mathbb{R}^n : p{\cdot}x \leq w\}$ where $p = (p_1, ..., p_n)$ with $p_i > 0$ for all $i, w \in \mathbb{R}, w > 0$

   is bounded. Extra credit: Show that it is also closed so that it is compact since $p \cdot x - w$ is a continuous function of x and sets $X = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ are closed if f is continuous.

3. Prove these classical mathematical inequalities by constrained optimization.

   (a) The arithmetic mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ of positive numbers $x_1, ..., x_n$ is always at least as large as their geometric mean $(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$. To do this solve the following problem for any $y > 0$:

$$\max_{(x_1,...,x_n)\in\mathbb{R}^n_+} (x_1 \cdot x_2 \cdot ...x_n)^{\frac{1}{n}}$$

   subject to

$$\sum_{i=1}^{n} x_i = y.$$

   i. Form the Lagrangean. Argue that an optimal solution must exist
   ii. Derive the first-order conditions necessary conditions.
   iii. Show that there is a single point that satisfies the necessary conditions. Since an optimal point exists, the solution to necessary conditions must be the optimum. Solve for the optimal x_i.

iv. Show that at the optimal choices, the two averages coincide. Otherwise, arithmetic is larger than geometric.

(b) Do the same comparison between arithmetic and harmonic means. The harmonic mean of $x_1, ..., x_n$ is $\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$. Follow the same steps as above.

4. Which of the following claims are true and which are false. For false statements, give a counterexample, for true ones, provide a proof.

   (a) The only functions that are both concave and convex on $\mathbb{R}^n$ are affine functions of the form $f(x) = c + b \cdot x$ for some vectors $b, c \in \mathbb{R}^n$.

   (b) The only functions that are both quasiconcave and quasiconvex are affine functions.

   (c) Minimizing the distance from a point to a plane in R^3 can be written as a problem of maximizing a concave function on a convex set.

   (d) If $f(x)$ is a concave function on $\mathbb{R}^n$, then for all increasing and concave $g(y)$, the composite function $g(f(x))$ is also concave. You may assume twice differentiability here if you want.

5. A sleepy father likes to eat, play with his children and sleep. Let $f \geq 0$ denote the amount of food that the father eats, $c \geq 0$ the amount of time with children and $s \geq 0$ the amount of sleep. His utility function is increasing in all three components and for simplicity, assume that it takes the form (with strictly positive α_i):

$$u(f, c, s) = \alpha_f \ln(f) + \alpha_c \ln c + \alpha_s \ln s.$$

   (a) Show that this is a strictly concave function on strictly positive vectors in R^3_{++}. Therefore any point satisfying the first-order conditions on any convex feasible set is a global maximum.

   (b) Unfortunately to get food, the father must work and working is away from either sleep or playing time with children. What is the budget constraint for the father if he has 24 hours of total time and the wage (in terms of units of food per hour) is $w > 0$?

   (c) What is the feasible set? Is it defined by some quasiconvex constraint functions?

   (d) Write the maximization problem, the Lagrangean and the first-order conditions for the problem.

   (e) Argue that the non-negativity constraints are not binding but the budget constraint is binding.

   (f) Solve for the unique point satisfying the necessary conditions. Are the conditions for Weierstrass' theorem satisfied? Think about the boundary of the feasible set. Does it matter that the utility function is not continuous on the boundary?