Mathematics for Economists: Lecture 8

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This lecture covers

1. Constrained optimization: examples
2. Optimization with many equality constraints
3. Optimization subject to inequality constraints
4. Concave programming

Figure: Consumer's problem on a budget line

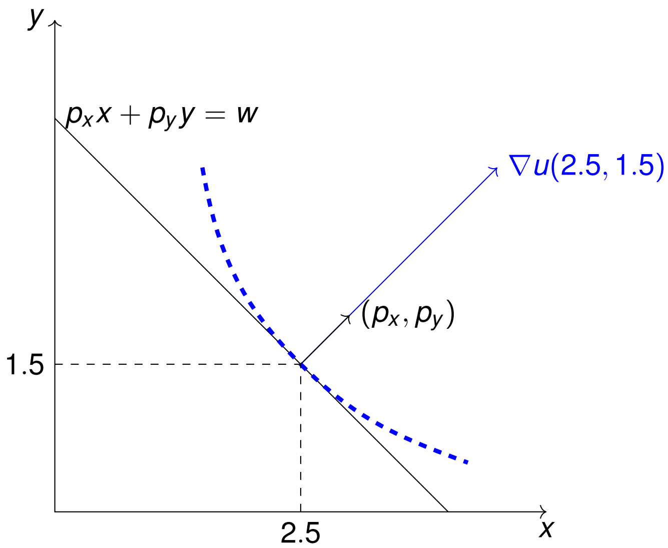
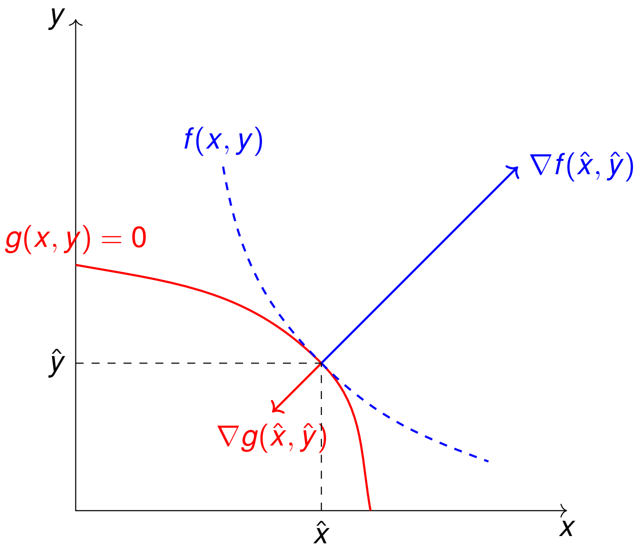


Figure: Single equality constraint



Optimization with a single equality constraint

Find the minima and maxima of $f(x, z) = x + z^2$ subject to constraints

$$x^2 + z^2 = 1$$

Form the Lagrangean

$$\mathcal{L}(x, z, \mu) = x + z^2 - \mu(x^2 + z^2 - 1)$$

Differentiate to get the first-order conditions (FOC):

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\mu x = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2z - 2\mu z = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 1 - x^2 - z^2 = 0 \qquad (3)$$

Optimization with a single equality constraint

- The second FOC gives:

$$z(2-2\mu)=0$$

- therefore either $z=0$, or $\mu=1$. Consider first the possibility that $z=0$. In that case, (3) implies that $x=\pm 1$. We get two critical points from (1):

$$\left(x=1, z=0, \mu=\tfrac{1}{2}\right) \text{ and } \left(x=-1, z=0, \mu=-\tfrac{1}{2}\right)$$

- If $\mu=1$, (1) implies that $x=\frac{1}{2}$. By substituting into (3) we get the critical points:

$$\left(x=\tfrac{1}{2}, z=\tfrac{\sqrt{3}}{2}, \mu=1\right) \text{ and } \left(x=\tfrac{1}{2}, z=-\tfrac{\sqrt{3}}{2}, \mu=1\right)$$

Optimization with a single equality constraint

- As a result, we have four critical points for the Lagrangean. Draw the constraint set and level curves for the objective function to get a guess of the classification of the critical points.
- By examining the bordered Hessian, we see that $(x = -1, z = 0, \mu = -\frac{1}{2})$ and $(x = 1, z = 0, \mu = \frac{1}{2})$ are local minima, and $\left(x = \frac{1}{2}, z = \pm\frac{\sqrt{3}}{2}, \mu = 1\right)$ are local maxima.
- Can you show the existence of a maximum? Which of the local maxima is the global maximum?

Optimization with multiple equality constraints

Consider next the case, where we have k equality constraints $g(x) = (g_1(x), ..., g_k(x) : \mathbb{R}^n \to \mathbb{R}^k$. In this case, we have the problem:

$$\max_x f(x)$$

$$\text{subject to } g_1(x) = 0,$$

$$g_2(x) = 0,$$

$$\vdots$$

$$g_k(x) = 0.$$

Form the Lagrangean now with k constraints as a function of $n + k$ variables:

$$\mathcal{L}(x, \mu_1, ..., \mu_k) = f(x) - \sum_{j=1}^{k} \mu_j g_j(x).$$

Optimization with multiple equality constraints

We can proceed exactly as before to use the linear approximations to characterize the feasible directions from $\hat{x}$ as $\{(x - \hat{x}) : Dg(\hat{x})(x - \hat{x})\} = 0$.

Since the objective function cannot increase at the optimum in any feasible direction, we have that

$$Df(\hat{x})(x - \hat{x}) = 0 \text{ whenever } Dg(\hat{x})(x - \hat{x}) = 0.$$

If $Dg(\hat{x})$ has full rank, then this is equivalent to requiring that $Df(\hat{x})$ and $Dg_j(\hat{x})$ must be linearly dependent. Since we assume that $Dg(\hat{x})$ has full rank, this means that there must exist $(\mu_1, ..., \mu_k)$ such that

$$\nabla f(\hat{x}) = \sum_{j=1}^{k} \mu_j \nabla g_j(\hat{x}).$$

Optimization with multiple equality constraints

Hence we can summarize the three necessary conditions for local maximum:

i) Gradient alignment: $\nabla f(\hat{x}) = \sum_{j=1}^{k} \mu_j \nabla g_j(\hat{x})$,

ii) Constraint holds: $g(\hat{x}) = 0$,

iii) Constraint qualification: $Dg_1(\hat{x}), ..., Dg_k(\hat{x})$ are linearly independent.

The first two can be achieved by requiring that $(\hat{x}, \hat{\mu}_1, ..., \hat{\mu}_k)$ be a critical point of the Lagrangean. The second-order conditions are based on bordered Hessian matrices as explained at the end of the previous subsection.

Optimization with multiple equality constraints: an example

- Consider the problem of maximizing

$$f(x, y, z) = xz + yz$$

subject to:

$$g_1(x, y, z) = y^2 + z^2 - 1$$
$$g_2(x, y, z) = xz - 3$$

1. Find the critical points of f subject to constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$.
2. How would you determine which of the critical points are local minima and which are local maxima? What is the bordered Hessian that you would use?

Optimization with multiple equality constraints: an example

1. Find first the critical points of the Lagrangean

$$\mathcal{L}(x, y, z, \mu_1, \mu_2) = xz + yz - \mu_1(y^2 + z^2 - 1) - \mu_2(xz - 3)$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = z - \mu_2 z = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial y} = z - 2\mu_1 y = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial z} = x + y - 2\mu_1 z - \mu_2 x = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = y^2 + z^2 - 1 = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} = xz - 3 = 0 \tag{8}$$

Optimization with multiple equality constraints: an example

We need to solve this system of equations to find the critical points. Start with (4), giving

$$z(1-\mu_2)=0, \Leftrightarrow \; z=0 \text{ or } \mu_2=1$$

If $z=0$, then (8) is not true for any x and as a result, we must have$z\neq 0$. Therefore, we can only have $\mu_2=1$ as a candidate solution. The second FOC (5) gives

$$y-2\mu_1 z=0, \;\Leftrightarrow y=\frac{z}{2\mu_1}.$$

Optimization with multiple equality constraints: an example

Plug in the solutions for y and μ_2 into (6) :

$$\frac{z}{2\mu_1} - 2\mu_1 z = 0 \iff z\left(\frac{1}{2\mu_1} - 2\mu_1\right) = 0.$$

We already know that $z \neq 0$, and therefore

$$\frac{1}{2\mu_1} - 2\mu_1 = 0 \iff 4\mu_1^2 = 1 \iff \mu_1 = \pm\frac{1}{2}$$

Optimization with multiple equality constraints: an example

We have now solved for possible Lagrange multipliers μ_1 ja μ_2, i.e. we have:

$$\mu_1 = \pm\frac{1}{2} \text{ and } \mu_2 = 1$$

Optimization with multiple equality constraints: an example

To get the values of the choice variables, plug in the values of the multipliers into (6) to get:

$$y = \pm z.$$

Substituting into (7), we get (by squaring):

$$2z^2 - 1 = 0 \ \Leftrightarrow z = \pm\frac{1}{\sqrt{2}}$$

The fifth FOC (8) gives:

$$x = \frac{3}{z},$$

or $x = 3\sqrt{2}$ if $z = \frac{1}{\sqrt{2}}$ and $x = -3\sqrt{2}$ if $z = -\frac{1}{\sqrt{2}}$. We have now found all that we need for the critical points of f subject to the constraints.

Optimization with multiple equality constraints: an example

If $z = \frac{1}{\sqrt{2}}$, then $x = 3\sqrt{2}$, $y = \pm z$. This yields two critical points (x, y, z):

$$1 : \left(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$2 : \left(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

If $z = -\frac{1}{\sqrt{2}}$, then $x = -3\sqrt{2}$, $y = \pm z$. This gives also two critical points (x, y, z):

$$3 : \left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$4 : \left(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Optimization with multiple equality constraints: an example

We know that for all critical points, $\mu_2 = 1$, and we can check the sign of μ_1 from FOC (5). After this, we have all the critical points of the problem as:

Critical points for the problem (x, y, z, μ_1, μ_2):

$$1: \left(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$$

$$2: \left(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right)$$

$$3: \left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$$

$$4: \left(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right)$$

Optimization with multiple equality constraints: an example

2. In order to determine which of the critical points are maxima or minima, we need to consider the bordered Hessian by bordering the Hessian of $\mathcal{L}$ with respect to (x, y, z), i.e. $D_x^2 L$ with the derivatives of the constraints $D_x g_1$ and $D_x g_2$.

$$D^2 L = \begin{bmatrix} 0 & Dg(\hat{x}) \\ [Dg(\hat{x})]^T & D_x^2 L \end{bmatrix}$$

Optimization with multiple equality constraints: an example

- With three dimensional choice variable, we know that $D_x^2 L$ is a $[3 \times 3]$ -matrix.
- With two constraints $Dg(\hat{x})$ is a $[2 \times 3]$ -matrix.
- Hence we have a (5×5) square matrix for our bordered Hessian:

$$D^2 L = \begin{bmatrix} 0 & 0 & g_{1x} & g_{1y} & h_{1z} \\ 0 & 0 & g_{2x} & g_{2y} & g_{2z} \\ g_{1x} & g_{2x} & L_{xx} & L_{xy} & L_{xz} \\ g_{1y} & g_{2y} & L_{yx} & L_{yy} & L_{yz} \\ g_{1z} & g_{2z} & L_{zx} & L_{zy} & L_{zz} \end{bmatrix},$$

Optimization with inequality constraints

The most important class of optimization problems in economics considers maximizing (or minimizing) an objective function subject to k inequality constraints.

In these problems, the feasible set X takes the form

$$X = \{x \in \mathbb{R}^n : h(x) \leq 0\},$$

where $h(x) \leq 0$ can written more fully as:

$$\begin{aligned} h_1(x_1, ..., x_n) &\leq 0 \\ \vdots \quad & \quad \vdots \\ h_k(x_1, ..., x_n) &\leq 0 \end{aligned}$$

Notice that we can incorporate equality constrains into these problems since $\{x \in \mathbb{R}^n : g(x) = 0\}$ is the same set as $\{x \in \mathbb{R}^n : g(x) \leq 0, \quad -g(x) \leq 0\}$.

Optimization with inequality constraints

- We shall concentrate on the first-order conditions for an optimum.
- Quite often, any point satisfying the first-order conditions is a global optimum if the derivative of the objective function at the point in question is non-zero.
- We say that an inequality constraint $h_j(x_1, ..., x_n) \leq 0$ is binding at $\hat{x}$ if $h_j(\hat{x}) = 0$.
- If $h_j(\hat{x}) < 0$, then we say that the constraint is not binding. A non-binding constraint does not restrict the feasible directions for small changes in $\hat{x}$. For binding constraints $h_j(\hat{x})$, the feasible directions Δx are given again by:

$$Dh_j(\hat{x})\Delta x \leq 0.$$

Optimization with inequality constraints

- Hence the binding constraints are just like the equality constraints that we discussed in the previous section.
- Non-binding constraints can be ignored. The problem in general is that we do not know a priori which constraints are binding and which are not.
- Let's write the Lagrangean function for the optimization problem as before:

$$\mathcal{L}(x, \lambda_1, ..., \lambda_k) = f(x) - \sum_{j=1}^{k} \lambda_j h_j(x).$$

Optimization with inequality constraints

- I have adopted the notation for the textbook to denote the Lagrange multipliers in inequality constrained problems by λ_j.
- If a constraint is not binding, it can be ignored in the problem.
- If it binds, then it cannot be ignored.
- But both of these cases are incorporated in the following complementary slackness condition. For all j, we have:

$$\lambda_j h_j(\hat{x}_1, ..., \hat{x}_n) = 0.$$

Optimization with inequality constraints

- This simply says that if $h_j(/hatx) < 0$, then $\lambda_j = 0$ and the constraint vanishes from the Lagrangean.
- If the constraint binds, then $h_j(\hat{x}) = 0$ and the complementary slackness is also satisfied.
- Based on these considerations, we formulate the first order conditions for $(\hat{x}, \hat{\lambda})$ as follows.
- We consider a point where the constraint qualification holds (i.e. the derivatives of the binding constraints are linearly independent so that we can use implicit function theorem).

Optimization with inequality constraints

- The first-order conditions for the problem also known as the Kuhn-Tucker or Karush-Kuhn-Tucker conditions for the problem are given by:

$$\frac{\partial \mathcal{L}}{\partial x_i}(\hat{x}, \hat{\lambda}) = \frac{\partial f}{\partial x_i}(\hat{x}) - \sum_{j=1}^{k} \hat{\lambda}_j \frac{\partial h_j}{\partial x_i}(\hat{x}) = 0 \text{ for all } i,$$

$$\hat{\lambda}_j h_j(\hat{x}) = 0 \text{ for all } j \in \{1, ..., k\},$$

$$\hat{\lambda}_j \geq 0 \text{ for all } j \in \{1, ..., k\},$$

$$h_j(\hat{x}) \leq 0 \text{ for all } j \in \{1, ..., k\}.$$

Optimization with inequality constraints

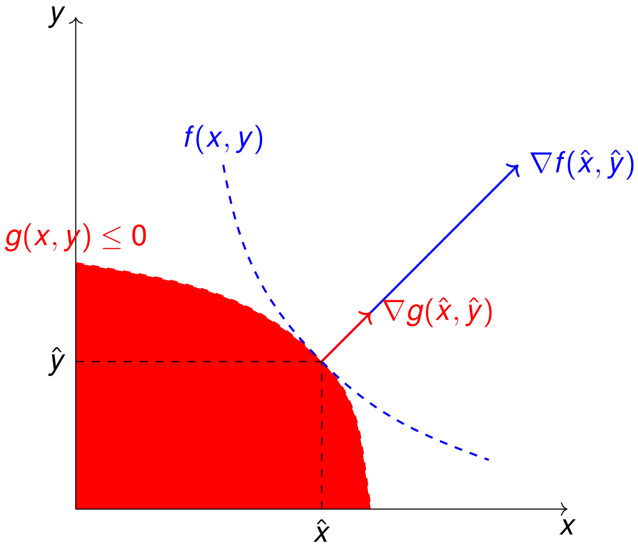
- Let me sum up: at the optimal point $\hat{x}$, we need

  i) the usual first-order condition for the Lagrangean with respect to the choice variables.

  ii) we need that $\hat{x}$ be feasible, i.e. $h_j(\hat{x}) \leq 0$ for all j,

  iii) the complementary slackness conditions, and the non-negativity of the multipliers.

Figure: Single inequality constraint



Optimization with inequality constraints

- We have not discussed the non-negativity of the multipliers yet, but it is easy to see why this must be true in the case of a single inequality constraint.
- Assume constraint qualification, i.e. $Dh(\hat{x}) \neq 0$By the first order conditions with respect to the x_i, we see that as before,

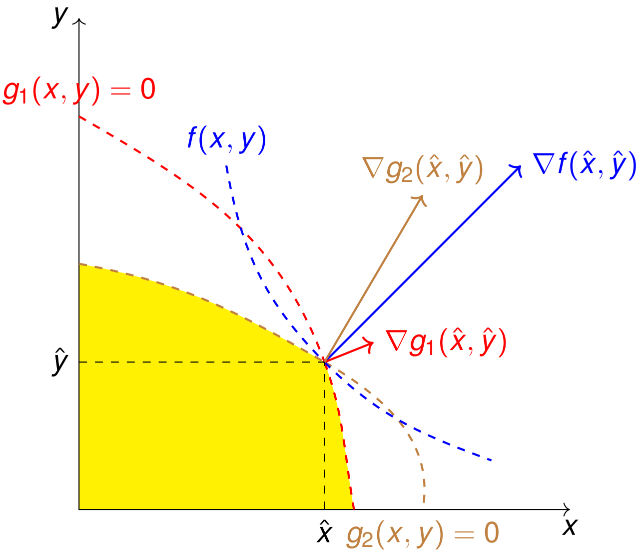
$$\nabla h(\hat{x}) = \lambda \nabla f(\hat{x}).$$

- If the multiplier was strictly negative at an optimal point $\hat{x}$, where the constraint binds, then

$$Dh(\hat{x}) \nabla f(\hat{x}) = \lambda \nabla h(\hat{x}) \cdot \nabla h(\hat{x}) \leq 0.$$

- Hence movement in the direction of the fastest increase of f is feasible and $\hat{x}$ cannot be an optimum unless $\nabla f(\hat{x}) = 0$. But in this case, $\lambda = 0$ since $\nabla h(\hat{x}) \neq 0$ by constraint qualification.
- The general case for the positive sign of the multipliers is proved using either separating hyperplane theorem or Farkas' Lemma and it is left for future studies.

Figure: Two inequality constraints



Concave programming

Consider i) maximization problems where the objective function is quasiconcave and ii) minimization problems where the objective function is quasiconvex.

For each of these cases, we assume that the constraint functions h_j are quasiconvex so that the feasible set that is given as the intersection of lower level sets of these functions is convex.

We are now ready to see why the first-order conditions are sufficient for maxima of quasiconcave functions with a non-vanishing derivative on a convex set.

Concave programming

Recall from Lecture 6 that a differentiable function f on a convex set X is quasiconcave if and only if for all $x, y \in X$:

$$f(y) \geq f(x) \Rightarrow Df(x)(y - x) \geq 0.$$

This implies the following (almost converse) result:

Proposition

Suppose $Df(x)$ is non-zero for all $x \in X$ and f is quasiconcave on X. Then $\hat{x}$ is a global maximum for f on X if $Df(\hat{x})(y - \hat{x}) \leq 0$ for all $y \in X$

Concave programming

Theorem

Suppose that f *is quasiconcave and* $Df(x) \neq 0$ *on a the convex set* $X = \{x \in \mathbb{R}^n : h_j(x) \leq 0 \text{ for } j \in \{1, ..., k\}\}$ *, where each* $h_j(x)$ *is a quasiconvex function. Then any point satisfying the first-order conditions is a global maximum for* f *on* X*.*

Concave programming

Proof. Write the first-order condition with respect to x as:

$$Df(\hat{x}) - \sum_{j=1}^{k} \hat{\lambda}_j Dh_j(\hat{x}) = 0. \tag{9}$$

Multiply on the right by $(y - \hat{x})$ to get

$$Df(\hat{x})(y - \hat{x}) - \sum_{j=1}^{k} \hat{\lambda}_j Dh_j(\hat{x})(y - \hat{x}) = 0. \tag{10}$$

Concave programming

For feasible directions for binding constraints, we have $Dh_j(\hat{x})(y - \hat{x}) \leq 0$ since each h_j is assumed to be quasiconvex. For nonbinding constraints, $\hat{\lambda}_j = 0$. Therefore since $\hat{\lambda}_j \geq 0$ for all j, we have

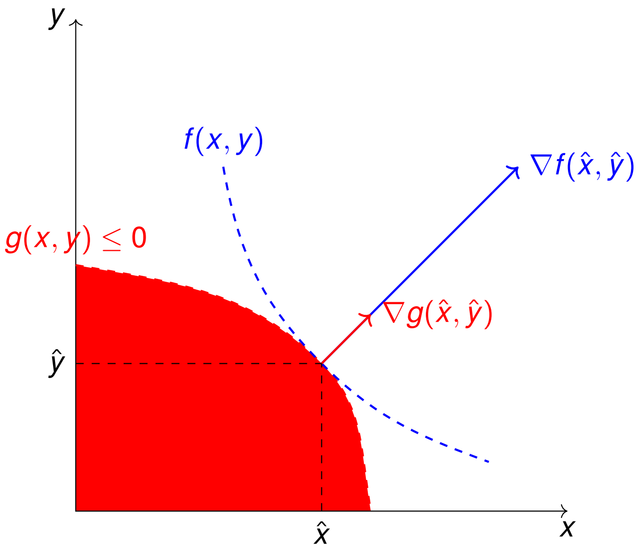
$$\hat{\lambda}_j Dh_j(\hat{x})(y - \hat{x}) \leq 0 \text{ for all } j.$$

Thus by equation (10), we see that

$$Df(\hat{x})(y - \hat{x}) \leq 0$$

for all feasible y. Therefore by the proposition above, $f(\hat{x}) \geq f(y)$ for all feasible y.

Figure: Single inequality constraint



Using Kuhn-Tucker conditions

Before getting into economic applications proper next week, let's conclude this section with a couple of numerical examples demonstrating how to find constrained maxima.

Maximize the objective function $f(x, y, z) = xyz + z$,
subject to

$$\begin{aligned} x^2 + y^2 + z &\leq 6 \\ x &\geq 0 \\ y &\geq 0 \\ z &\geq 0. \end{aligned}$$

1. Find the points that satisfy the first-order condition
2. Investigate whether the constraint $x^2 + y^2 + z \leq 6$ binds at optimum.
3. Find a point satisfying the first order conditions with $x = 0$.

Using Kuhn-Tucker conditions

Let's do this mechanically and think about what we do only afterwards.

From the Lagrangean:

$$\mathcal{L}(x, y, z, \lambda_i) = xyz + z - \lambda_1 \left[x^2 + y^2 + z - 6\right] + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

Using Kuhn-Tucker conditions

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = yz - 2\lambda_1 x + \lambda_2 = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial y} = xz - \lambda_1 + \lambda_3 = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial z} = xy + 1 - \lambda_1 + \lambda_4 = 0 \tag{13}$$

$$\lambda_1 \left[x^2 + y^2 + z - 6\right] = 0 \tag{14}$$

$$\lambda_2 x = 0, \lambda_3 y = 0, \lambda_4 z = 0 \tag{15}$$

$$x^2 + y^2 + z \leq 6 \tag{16}$$

$$-x \leq 0, -y \leq 0, -z \leq 0 \tag{17}$$

$$\lambda_i \geq 0 \quad i \in \{1, 2, 3, 4\} \tag{18}$$

Using Kuhn-Tucker conditions

If at optimum, $\lambda_1 = 0$, i.e. the first constraint is not binding, we get from (13):

$$xy + 1 + \lambda_4 = 0$$

This is not possible for any feasible (x, y), since $\lambda_4 \geq 0$, implying that either x or y must be negative. By the non-negativity constraints, we conclude that

$$\lambda_1 > 0,$$

and $x^2 + y^2 + z \leq 6$ binds at the optimum.

Using Kuhn-Tucker conditions

Find a critical point with $x = 0$:

$$\frac{\partial \mathcal{L}}{\partial x} = yz + \lambda_2 = 0 \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -\lambda_1 + \lambda_3 = 0 \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 1 - \lambda_1 + \lambda_4 = 0 \tag{21}$$

$$\lambda_1 \left[y^2 + z - 6 \right] = 0 \tag{22}$$

$$\lambda_2 x = 0, \lambda_3 y = 0, \lambda_4 z = 0 \tag{23}$$

$$y^2 + z \leq 6 \tag{24}$$

$$-x \leq 0, -y \leq 0, -z \leq 0 \tag{25}$$

$$\lambda_i \geq 0 \quad i \in \{1, 2, 3, 4\} \tag{26}$$

Using Kuhn-Tucker conditions

- By the previous part, we know that $\lambda_1 > 0$.
- The second FOC gives $\lambda_3 = \lambda_1$.
- By non-negativity of y we get $y = 0$.
- The first FOC then requires that $\lambda_2 = 0$, and therefore constraint (24) must bind at optimum.
- This yields $z = 6$. We have therefore found the point $(x = 0, y = 0, z = 6)$ and $(\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0)$ satisfying the first-order conditions and $x = 0$.

Using Kuhn-Tucker conditions

- Can you show that this is a local maximum (do not compute the bordered Hessian but argue directly using the directional derivative in all feasible directions)?
- Can you find another local maximum that gives a higher value to the objective function? (Use symmetry to argue that at any local maximum, $\hat{x} = \hat{y}$ reducing the problem essentially to a bivariate problem. look for interior solutions).

Using Kuhn-Tucker conditions

The next problem is to find a maximum in a problem that is really a consumer optimization problem without the economics terminology.

Maximize

$$f(x, y) = \alpha x + \sqrt{y}$$

subject to

$$\begin{aligned} px + y &\leq 1, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

Let's assume that $p > 0$ and let's find interior solutions, i.e. $(\hat{x}, \hat{y}) > 0$. Are there other kinds of solutions?

Using Kuhn-Tucker conditions

Form the Lagrangean:

$$\mathcal{L}(x, y, \lambda_i) = \alpha x + \sqrt{y} - \lambda_1 [px + y - 1] + \lambda_2 x + \lambda_3 y$$

First-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha - \lambda_1 p + \lambda_2 = 0 \tag{27}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2} y^{-\frac{1}{2}} - \lambda_1 + \lambda_3 = 0 \tag{28}$$

$$\lambda_1 [px + y - 1] = 0 \tag{29}$$

$$\lambda_2 x = \lambda_3 y = 0 \tag{30}$$

$$px + y \leq 1 \tag{31}$$

$$-x \leq 0, -y \leq 0 \tag{32}$$

$$\lambda_i \geq 0 \quad i \in \{1, 2, 3\} \tag{33}$$

Using Kuhn-Tucker conditions

If $x, y > 0$, we have $\lambda_2, \lambda_3 = 0$:
First-order conditions give:

$$\lambda_1 = \frac{1}{2} y^{-\frac{1}{2}} \Leftrightarrow \lambda_1 > 0$$

Substitute the obtained multiplier into (27) to get:

$$\alpha - \frac{1}{2} y^{-\frac{1}{2}} p = 0 \Leftrightarrow y^* = \frac{p^2}{4\alpha^2}$$

Using Kuhn-Tucker conditions

Since $\lambda_1 > 0$, we have $px + y = 1$ at optimum. This gives:

$$x^* = \frac{4\alpha^2 - p^2}{4p\alpha^2}$$

Note that this solution is valid only if $2\alpha \geq p$. Constraint qualification holds since the derivative of the binding constraint is nonzero at optimum: $D(h_1(x, y)) = \begin{bmatrix} p & 1 \end{bmatrix}$. If $p > 2\alpha$, then the optimum is a corner solution. From (28), we see that at any optimum, $y > 0$. Therefore the only other possibility is that $(\hat{x}, \hat{y})$ = (0,1). What is the value of λ_2 in this case?