Mathematics for Economists

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Solutions to the problem set 3:

Question 1:

a)

y: vector of endogenous variables

x: vector of exogenous variables

We will use the implicit function theorem to obtain the partial derivative of y with respect to x. The first thing that we have to do is to check if the equations hold at the mentioned point, which they do.

$$f_1(y_1, y_2; \alpha, \beta, q) = y_1 + \alpha y_2^2 - q = 0$$

$$f_2(y_1, y_2; \alpha, \beta, q) = \beta y_1^2 + y_2 - q = 0$$

$$f_1(1,1; 1,1,2) = 1 + 1 - 2 = 0$$

$$f_2(1,1; 1,1,2) = 1 + 1 - 2 = 0$$

According to the implicit function theorem:

$$D_y f(\hat{y}, \hat{x}) dy + D_x f(\hat{y}, \hat{x}) dx = 0$$

$$\Rightarrow dy = -\left(D_y f(\hat{y}, \hat{x})\right)^{-1} D_x f(\hat{y}, \hat{x}) dx$$

$$D_y f(\hat{y}, \hat{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 2\alpha y_2 \\ 2\beta y_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

And the determinant of this matrix is not equal to zero so we have the second condition of the implicit function theorem.

$$D_x f(\hat{y}, \hat{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial q} \\ \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial q} \end{bmatrix} = \begin{bmatrix} y_2^2 & 0 & -1 \\ 0 & y_1^2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$dy = \left(-\frac{1}{3}\right) \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} dx = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} dx$$

b) Matrix of the partial derivatives:

$$\begin{bmatrix} \frac{\partial y_1}{\partial \alpha} & \frac{\partial y_1}{\partial \beta} & \frac{\partial y_1}{\partial q} \\ \frac{\partial y_2}{\partial \alpha} & \frac{\partial y_2}{\partial \beta} & \frac{\partial y_2}{\partial q} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Question 2:

a) The optimization problem of the firm:

$$\max_{p_1, p_2} \; p_1 . q_1(p_1, p_2, x_1) + p_2 . q_2(p_1, p_2, x_1) - c_1 q_1^2 - c_2 q_2^2$$

b) The assumptions are:
Demand function is always a decreasing function of the price, so:

$$\frac{\partial q_1}{\partial p_1} \leq 0 \; and \; \frac{\partial q_2}{\partial p_2} \leq 0$$

We also know that when the price for one of the goods increases the demand for the other one will also increase, so:

$$\frac{\partial q_1}{\partial p_2} \geq 0 \; and \; \frac{\partial q_2}{\partial p_1} \geq 0$$

c) Considering the optimization problem as F:

$$F = \; p_1 . q_1(p_1, p_2, x_1) + p_2 . q_2(p_1, p_2, x_1) - c_1 q_1^2 - c_2 q_2^2$$

So the first order condition is:

$$\nabla F = 0 \Rightarrow \begin{bmatrix} \frac{\partial F}{\partial p_1} \\ \frac{\partial F}{\partial p_2} \end{bmatrix} = 0$$

d) The demand functions are:

$$q_1 = x_1 + \alpha_1 p_1 + \beta_1 p_2$$
$$q_2 = x_2 + \alpha_2 p_1 + \beta_2 p_2$$
$$F = p_1 q_1 + p_2 q_2 - c_1 q_1^2 - c_2 q_2^2 \Rightarrow$$

$$\frac{\partial F}{\partial p_1} = q_1 + \alpha_1 p_1 + \alpha_2 p_2 - 2\alpha_1 c_1 q_1 - 2\alpha_2 c_2 q_2$$
$$\frac{\partial F}{\partial p_2} = \beta_1 p_1 + q_2 + \beta_2 p_2 - 2\beta_1 c_1 q_1 - 2\beta_2 c_2 q_2$$

$$\underbrace{\begin{bmatrix} 2\alpha_1 - 2c_1\alpha_1^2 - 2c_2\alpha_2^2 & \beta_1 + \alpha_2 - 2\alpha_1\beta_1 c_1 - 2\alpha_2\beta_2 c_2 \\ \beta_1 + \alpha_2 - 2\alpha_1\beta_1 c_1 - 2\alpha_2\beta_2 c_2 & 2\beta_2 - 2c_1\beta_1^2 - 2c_2\beta_2^2 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} =$$

$$= \begin{bmatrix} -x_1 + 2\alpha_1 c_1 x_1 + 2\alpha_2 c_2 x_2 \\ -x_2 + 2\beta_1 c_1 x_1 + 2\beta_2 c_2 x_2 \end{bmatrix}$$

Finally

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} =$$

$$= \left(\frac{1}{\det(A)}\right) \begin{bmatrix} 2\beta_2 - 2c_1\beta_1^2 - 2c_2\beta_2^2 & -\beta_1 - \alpha_2 + 2\alpha_1\beta_1 c_1 + 2\alpha_2\beta_2 c_2 \\ -\beta_1 - \alpha_2 + 2\alpha_1\beta_1 c_1 + 2\alpha_2\beta_2 c_2 & 2\alpha_1 - 2c_1\alpha_1^2 - 2c_2\alpha_2^2 \end{bmatrix} \begin{bmatrix} -x_1 + 2\alpha_1 c_1 x_1 + 2\alpha_2 c_2 x_2 \\ -x_2 + 2\beta_1 c_1 x_1 + 2\beta_2 c_2 x_2 \end{bmatrix}$$

And about the coefficients, we go back to the part b:

$$\frac{\partial q_1}{\partial p_1} \leq 0 \Rightarrow \alpha_1 \leq 0$$

$$\frac{\partial q_1}{\partial p_2} \geq 0 \Rightarrow \beta_1 \geq 0$$

$$\frac{\partial q_2}{\partial p_2} \leq 0 \Rightarrow \beta_2 \leq 0$$

$$\frac{\partial q_2}{\partial p_1} \geq 0 \Rightarrow \alpha_2 \geq 0$$

Moreover, we only have positive demands for any amount of the price values, so:

$$x_1, x_2 \geq 0$$

Question 3:

a)

i)

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

Using the quadratic form:

$$x_1^2 + 6x_1x_2 + 5x_2^2$$

$$\mathrm{a} = 1 > 0$$

$$\det(A) = -4 < 0$$

So the matrix A is indefinite.

ii)

$$B = \begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$$

$$a = -1 < 0$$

$$\det(B) = 4 > 0$$

So B is negative definite.

$$C = \begin{bmatrix} -1 & -2 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & -9 \end{bmatrix}$$

C is not symmetric and it has diagonal elements of different sign, so it is indefinite.

b)

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$$

$$a = 1 > 0$$

$$\det(A) = 9 - b^2 > 0 \rightarrow b^2 < 9 \rightarrow b \in (-3,3)$$

c)

$$f(x,y) = \frac{1}{2}(x^2 + 2bxy + 9y^2)$$

$$\nabla f(x,y) = (x + by, 9y + bx) = 0$$

It yields only the trivial solution $(0,0)$. We also have:

$$Hf(x,y) = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$$

But we proved before that H is positive definite, so the critical point (0,0) is a global minimizer of $f(x,y)$.

Question 4:

a) $f(x) = x^3$

$$f'(x) = 3x^2 \rightarrow f''(x) = 6x$$

$$\begin{cases} f''(x) > 0 & if\ x > 0 \\ f''(x) < 0 & if\ x < 0 \end{cases}$$

So f is concave for negative values of x.

b) $f(x,y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$

$$\nabla f(x,y) = (\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}})$$

$$H = \begin{bmatrix} \frac{-\frac{1}{4}}{x^{\frac{3}{2}}} & 0 \\ 0 & \frac{-\frac{1}{4}}{y^{\frac{3}{2}}} \end{bmatrix}$$

H is negative definite in it’s domain, so the function f is concave.

c) $f(x,y) = -x^2 - y^2 + 3xy$

$$\nabla f(x,y) = (-2x + 3y, -2y + 3x)$$

$$H = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$$

Matrix H is indefinite so the function f is not concave nor convex.

Question 5:

$$f(x_1, x_2) = Ax_1^{\alpha_1}x_2^{\alpha_2}$$

$$\nabla f(x, y) = (A\alpha_1 x_1^{\alpha_1 - 1}x_2^{\alpha_2}, A\alpha_2 x_2^{\alpha_2 - 1}x_1^{\alpha_1})$$

$$H = \begin{bmatrix} A\alpha_1(\alpha_1 - 1)x_1^{\alpha_1 - 2}x_2^{\alpha_2} & A\alpha_1\alpha_2 x_1^{\alpha_1 - 1}x_2^{\alpha_2 - 1} \\ A\alpha_1\alpha_2 x_1^{\alpha_1 - 1}x_2^{\alpha_2 - 1} & A\alpha_2(\alpha_2 - 1)x_2^{\alpha_2 - 2}x_1^{\alpha_1} \end{bmatrix}$$

a) For matrix H to be negative definite we should have two conditions:
   1- For the matrix to be concave:

$$A\alpha_1(\alpha_1 - 1)x_1^{\alpha_1 - 2}x_2^{\alpha_2} \leq 0 \rightarrow \alpha_1 - 1 \leq 0 \rightarrow \alpha_1 \leq 1$$

$$A\alpha_2(\alpha_2 - 1)x_1^{\alpha_1}x_2^{\alpha_2 - 2} \leq 0 \rightarrow \alpha_2 - 1 \leq 0 \rightarrow \alpha_2 \leq 1$$

And finally

$$\det(H) = A\alpha_1(\alpha_1 - 1)x_1^{\alpha_1 - 2}x_2^{\alpha_2}.A\alpha_2(\alpha_2 - 1)x_2^{\alpha_2 - 2}x_1^{\alpha_1} - (A\alpha_1\alpha_2 x_1^{\alpha_1 - 1}x_2^{\alpha_2 - 1})^2 \geq 0$$

$$\rightarrow A^2\alpha_1\alpha_2(\alpha_1 - 1)(\alpha_2 - 1)x_1^{2\alpha_1 - 2}x_2^{2\alpha_2 - 2} - A^2(\alpha_1\alpha_2)^2 x_1^{2\alpha_1 - 2}x_2^{2\alpha_2 - 2} \geq 0$$

$$\alpha_1\alpha_2(\alpha_1 - 1)(\alpha_2 - 1) - (\alpha_1\alpha_2)^2 \geq 0$$

$$\alpha_1\alpha_2(\alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1 - \alpha_1\alpha_2) \geq 0$$

$$\alpha_1 + \alpha_2 \leq 1$$

So for the function f to be concave we should have:

$$\alpha_1 \leq 1, \alpha_2 \leq 1, \alpha_1 + \alpha_2 \leq 1$$

For having strictly concave we should have the strict form of the above constraints:

$$\alpha_1 < 1, \alpha_2 < 1, \alpha_1 + \alpha_2 < 1$$

b) First we should form the bordered hessian Matrix:

$$\hat{H} = \begin{bmatrix} 0 & A\alpha_1 x_1^{\alpha_1 - 1}x_2^{\alpha_2} & A\alpha_2 x_2^{\alpha_2 - 1}x_1^{\alpha_1} \\ A\alpha_1 x_1^{\alpha_1 - 1}x_2^{\alpha_2} & A\alpha_1(\alpha_1 - 1)x_1^{\alpha_1 - 2}x_2^{\alpha_2} & A\alpha_1\alpha_2 x_1^{\alpha_1 - 1}x_2^{\alpha_2 - 1} \\ A\alpha_2 x_2^{\alpha_2 - 1}x_1^{\alpha_1} & A\alpha_1\alpha_2 x_1^{\alpha_1 - 1}x_2^{\alpha_2 - 1} & A\alpha_2(\alpha_2 - 1)x_2^{\alpha_2 - 2}x_1^{\alpha_1} \end{bmatrix}$$

We know that for a function to be quasiconcave we should have:

$$|\hat{H}_1| = det\begin{bmatrix} 0 & A\alpha_1 x_1^{\alpha_1 - 1}x_2^{\alpha_2} \\ A\alpha_1 x_1^{\alpha_1 - 1}x_2^{\alpha_2} & A\alpha_1(\alpha_1 - 1)x_1^{\alpha_1 - 2}x_2^{\alpha_2} \end{bmatrix} \leq 0$$

And

$$\left|\hat{H}_2\right| = det\begin{bmatrix} 0 & A\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2} & A\alpha_2 x_2^{\alpha_2-1}x_1^{\alpha_1} \\ A\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2} & A\alpha_1(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} & A\alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} \\ A\alpha_2 x_2^{\alpha_2-1}x_1^{\alpha_1} & A\alpha_1\alpha_2 x_1^{\alpha_1-1}x_2^{\alpha_2-1} & A\alpha_2(\alpha_2-1)x_2^{\alpha_2-2}x_1^{\alpha_1} \end{bmatrix} \geq 0$$

And in the most general case:

$$|\hat{H}_n|\begin{cases} \leq 0 \; if \; n \; is \; odd \\ \geq 0 \; if \; n \; is \; even \end{cases}$$

For the strict case we should have strict inequalities.

For the first condition, we have:

$$-A^2\alpha_1^2 x_1^{2\alpha_1-2}x_2^{2\alpha_2} < 0$$

So it holds for any value of α_1, α_2, and for the second one:

$$-\alpha_1 x_1 x_2^2(\alpha_1\alpha_2(\alpha_2-1)x_1^3x_2^2 - \alpha_1\alpha_2^2x_1^3x_2^3) +$$

$$(\alpha_2 x_2 x_1^2)(\alpha_1^2\alpha_2 x_1^2 x_2^3 - \alpha_1\alpha_2(\alpha_1-1)x_1^2x_2^3) =$$

After simplifying the equations we have:

$$-\alpha_1(\alpha_2 - 1 - \alpha_2) + \alpha_2(\alpha_1 - \alpha_1 + 1) = \alpha_1 + \alpha_2 \geq 0$$

And we know that α_1, α_2 are always positive so this condition is also valid for any value of α_1, α_2. Consequently, this function is quasiconcave for all positive alphas.

But there is an easier way to verify that the function is quasiconcave. Assume function g to be:

$$g(x_1, x_2) = (Ax_1^{\alpha_1}x_2^{\alpha_2})^{\frac{1}{\alpha_1+\alpha_2}} = A'x_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}}x_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}$$

From part a we know that this function is concave because it has the necessary conditions:

$$\frac{\alpha_1}{\alpha_1+\alpha_2} \leq 1, \frac{\alpha_2}{\alpha_1+\alpha_2} \leq 1, \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2} = 1$$

So $g(x_1, x_2)$ is a quasiconcave function. Now we define strictly increasing function h as:

$$h(x) = x^{\alpha_1+\alpha_2}$$

and we know that function f can stated as:

$$f(x_1, x_2) = h(g(x_1, x_2))$$

Now because g is quasiconcave function and h is strictly increasing function, so the function f is also a quasiconcave function for all positive values of α_1, α_2.