Mathematics for Economists: Lecture 9

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This lecture covers

1. Utility maximization problem (UMP): generalities
2. UMP: Constant elasticity of substitution
3. UMP: Cobb-Douglas and Stone-Geary
4. UMP: Quasilinear utility
5. Expenditure minimization
6. Cost minimization

Utility maximization problem (UMP)

- A consumer allocates her budget of $w > 0$ to n goods.
- Her consumption vector is an element of the positive orthant of the n Euclidean space $X = \{x \in \mathbb{R}^n_+\}$.
- We assume that the consumer has a continuous utility function $u(x)$ defined on X.
- Economic scarcity is present through the budget constraint:

$$p \cdot x \leq w \text{ or } \sum_{i=1}^{n} p_i x_i \leq w,$$

where $p = (p_1, ..., p_n) > 0$ is the vector of strictly positive prices for the goods.

Utility maximization problem (UMP)

Maximize

$$u(x_1, ..., x_n)$$

subject to

$$\sum_{i=1}^{n} p_i x_i \leq w,$$
$$x_i \geq 0 \text{ for all } i.$$

Alternatively. subject to

$$\sum_{i=1}^{n} p_i x_i - w \leq 0,$$
$$-x_i \leq 0 \text{ for all } i,$$

Utility maximization problem (UMP)

- To see that the feasible set is bounded, let $p^{min} = \min_j p_j$ (i.e. one of the smallest prices p_j).
- Then we know that for all feasible x, we have $p_i x_i \leq w$ for all i since $x_i \geq 0$ and $p_i > 0$ for all i.
- Therefore for all feasible x, $x_i \leq \frac{w}{p^{min}}$ for all i so that the feasible set is bounded since $0 \leq x_i \leq \frac{w}{p^{min}}$ for all i.

Utility maximization problem (UMP)

- To see that the feasible set is closed, we need to show that all limit points of the feasible belong to the feasible set.
- We show this by arguing that when y is not in the feasible set, it is not a limit point.
- If y is not feasible, then either $y_i < 0$ for some i or $\sum_i p_i y_i > w$.
- In both cases all points in a small enough neighborhood of y in infeasible. In the first case, $B^{\varepsilon}(y)$ with $\varepsilon < -\min_i y_i$, in the second, $\varepsilon < \frac{\sum_i p_i y_i - w}{\max_i p_i}$.
- Weiertrass' theorem guarantees that a maximum exists. The solution is called the Marshallian demand (demand as a function of prices and income).

UMP: Lagrangean

- Since the constraint functions are linear, the feasible set is convex.
- If u is strictly increasing (as we usually assume) and quasiconcove, then the first order Kuhn-Tucker conditions are necessary and sufficient for optimum.
- In words, whenever we find a point satisfying the K-T conditions, we have solved the problem.
- Lagrangean:

$$\mathcal{L}(x, \lambda) = u(x) - \lambda_0 \left[\sum_{i=1}^{n} p_i x_i - w \right] + \sum_{i=1}^{n} \lambda_i x_i$$

UMP: K-T conditions

- The first-order K-T conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial u(x)}{\partial x_i} - \lambda_0 p_i + \lambda_i = 0 \text{ for all } i, \tag{1}$$

$$\lambda_0 \left[\sum_{i=1}^{n} p_i x_i - w \right] = 0, \tag{2}$$

$$\lambda_i x_i = 0 \text{ for all } i, \tag{3}$$

$$\sum_{i=1}^{n} p_i x_i - w \leq 0, \tag{4}$$

$$-x_i \leq 0 \text{ for all } i, \tag{5}$$

$$\lambda_i \geq 0 \quad i \in \{0, 1, ..., n\}. \tag{6}$$

UMP: Simplifying the K-T conditions

- If the utility function has a strictly positive partial derivative for some x_i at the optimum, then the budget constraint must bind and $\lambda_0 > 0$.
- This follows immediately from the first line of the K-T conditions.
- For the other inequality constraints, consider the partial derivatives at $x \in X$with $x_i \to 0$ for some i.
- If

$$\lim_{x_i \to 0} \frac{\partial u(x)}{\partial x_i} = \infty,$$

  then the first line of the K-T conditions implies that at optimum $x_i > 0$.
- If this is true for all i, then we can ignore the non-negativity constraints and we are effectively back to a problem with a single equality constraint.
- If $\frac{\partial u(x)}{\partial x_i} < \infty$ for $x = (x_i, x_{-i}) = (0, x_{-i})$, then we must also consider corner solutions where $x_i = 0$ at optimum.

UMP: Interior solutions to K-T conditions

- For interior solutions $x_i > 0$ for all i, we get from the first equation by eliminating λ the familiar condition:

$$\frac{\frac{\partial u(x)}{\partial x_i}}{\frac{\partial u(x)}{\partial x_k}} = \frac{p_i}{p_k}. \tag{7}$$

- This is of course the familiar requirement that $MRS_{x_i,x_k} = \frac{p_i}{p_k}$ that we saw in Principles of Economics 1.
- Now we see that the same condition extends for many goods and the economic intuition is exactly the same.
- The price ratio gives the marginal rate of transformation between the different goods and at an interior optimum, that rate must coincide with the marginal rate of substitution.

UMP: Interior solutions to K-T conditions

- By multiplying these equations by $p_k \frac{\partial u(x)}{\partial x_k}$, we can write the first order conditions for an interior solution as:

$$p_k \frac{\partial u(x)}{\partial x_1} - p_1 \frac{\partial u(x)}{\partial x_k} = 0 \text{ for all } k, \sum_{i=1}^{n} p_i x_i - w = 0. \quad (8)$$

- In this equation system, we have n endogenous variables $x_1, ..., x_n$ and $n+1$ exogenous variables $p_1, ..., p_n, w$.
- We want to examine the comparative statics of $x(p,w)$, for example $\frac{\partial x_i(p,w)}{\partial p_i}$, $\frac{\partial x_i(p,w)}{\partial p_j}$ and $\frac{\partial x_i(p,w)}{\partial w}$.
- In words, what happens to the demand for one good when its own price changes, when other goods prices change and when income changes.
- In the next lecture, we'll do this via duality between UMP and expenditure minimization. Here, tackle easy cases where the optimum can be solved explicitly.

UMP: Constant elasticity of substitution

- In some cases, the functional form allows for explicit solution.
- We start with the constant elasticity of substitution utility function with two goods $x, y \in \mathbb{R}$.

$$u(x, y) = (a_x x^\rho + a_y y^\rho)^{\frac{1}{\rho}},$$

  for $\rho < 1 \rho \neq 0$ and $a_x, a_y > 0$.
- You have already shown in problem sets that functions of this type are quasiconcave for $\rho > 0$.
- Please show that the function is quasiconcave also for $\rho < 0$. Be careful here since raising to a negative power is not an increasing function.

UMP: Constant elasticity of substitution

- Compute the marginal utility for each x_i:

$$\frac{\partial u(x,y)}{\partial x} = \rho a_x x^{\rho-1} \frac{1}{\rho}(a_x x^\rho + a_y y^\rho)^{\frac{1}{\rho}-1}.$$

$$\frac{\partial u(x,y)}{\partial y} = \rho a_y y^{\rho-1} \frac{1}{\rho}(a_x x^\rho + a_y y^\rho)^{\frac{1}{\rho}-1}.$$

- Note that since $\rho < 1$, we have $\frac{\partial u(x,y)}{\partial x} > 0$, $\frac{\partial u(x,y)}{\partial y} > 0$, and

$$\lim_{x\to 0} \frac{\partial u(x,y)}{\partial x} = \lim_{y\to 0} \frac{\partial u(x,y)}{\partial y} = \infty.$$

- Hence budget constraint binds and interior solution
- Feasible set is convex, the objective function is quasiconcave with a non-vanishing derivative, the first order conditions are also sufficient

UMP: Constant elasticity of substitution

- Hence the K-T conditions require simply that for all i, k:

$$\frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} = \frac{p_x}{p_y},$$

and the budget constraint holds as an equality:

$$p_x x + p_y y = w.$$

- Hence we have that

$$\frac{a_x x^{\rho-1}}{a_y y^{\rho-1}} = \frac{p_x}{p_y},$$

or

$$\frac{x}{y} = \left(\frac{a_y p_x}{a_x p_y}\right)^{\frac{1}{\rho-1}},$$

or

UMP: Constant elasticity of substitution

- Substituting into the budget constraint, we get:

$$p_x x + p_y x \left(\frac{a_y p_x}{a_x p_y}\right)^{\frac{1}{1-\rho}} = w.$$

- We can solve for x_1 to get

$$x = \frac{w}{p_x + p_y \left(\frac{a_y p_x}{a_x p_y}\right)^{\frac{1}{1-\rho}}}.$$

- Simplifying the expression a bit, we get

$$x = \frac{w\left(\frac{p_x}{a_x}\right)^{\frac{1}{\rho-1}}}{(a_x)^{\frac{1}{1-\rho}}(p_x)^{\frac{\rho}{\rho-1}} + (a_y)^{\frac{1}{1-\rho}}(p_y)^{\frac{\rho}{\rho-1}}}, \qquad y = \frac{w\left(\frac{p_y}{a_y}\right)^{\frac{1}{\rho-1}}}{(a_x)^{\frac{1}{1-\rho}}(p_x)^{\frac{\rho}{\rho-1}} + (a_y)^{\frac{1}{1-\rho}}(p_y)^{\frac{\rho}{\rho-1}}}.$$

UMP: Constant elasticity of substitution

- Let $r = \frac{\rho}{\rho - 1}$. Then we have a bit more neatly:

$$x = \frac{w(\frac{p_x}{a_x})^{r-1}}{a_x^{1-r} p_x^r + a_y^{1-r} p_y^r}, \qquad y = \frac{w(\frac{p_y}{a_y})^{r-1}}{a_x^{1-r} p_x^r + a_y^{1-r} p_y^r}.$$

- For the case where $a_x = a_y$, this simplifies further to:

$$x(p_x, p_y, w) = \frac{w p_x^{r-1}}{p_x^r + p_y^r}, \qquad y(p_x, p_y, w) = \frac{w p_y^{r-1}}{p_x^r + p_y^r}.$$

- Exercise: Compute the comparative statics for $x(p_x, p_y, w), y(p_x, p_y, w)$ in the exogenous variables. What happens when $r \to 1$ and $r \to -\infty$?

UMP: Constant elasticity of substitution

- You will see in further studies the case with n goods $x = (x_1, ..., x_n)$ at prices $p = (p_1, ..., p_n)$. With equal coefficients $a_i = a_j$ for all i, j, the optimal demands are:

$$x_i(p, w) = \frac{wp_x^{r-1}}{\sum_{i=1}^{n} p_i^r}.$$

- The term $\sum_{i=1}^{n} p_i^r$ is called the CES price aggregator. You will see CES functions in international trade, endogenous growth, production theory and industrial organization.

UMP: Constant elasticity of substitution

- To see where the name comes from, go back to
$$\frac{x}{y} = \left(\frac{a_y p_x}{a_x p_y}\right)^{\frac{1}{\rho-1}}.$$
- If you denote $\frac{x}{y} = z, \frac{p_x}{p_y}) = q$, you have
$$z = cq^{\frac{1}{\rho-1}}.$$
- Hence $\frac{1}{\rho-1}$ is the elasticity of z with respect to q. The higher $\frac{1}{\rho-1}$, the more substitutable the products are.

UMP: Cobb-Douglas utility function

- In Problem set 1, you showed that as $\rho \to 0$, the CES -function converges to the Cobb-Douglas utility function $u(x) = x^{a_x} y^{a_y}$.
- We can take $a_x + a_y = 1$ and denote $a_x = \alpha, a_y = 1 - \alpha$ since raising to the power of $a_x + a_y$ is a strictly increasing function.
- Both marginal utilities are strictly positive and

$$\lim_{x \to 0} \frac{\partial u(x, y)}{\partial x} = \lim_{y \to 0} \frac{\partial u(x, y)}{\partial y} = \infty.$$

UMP: Cobb-Douglas utility function

- Hence interior solution and budget constraint binds.
- The requirement that $MRS_{x,y} = \frac{p_x}{p_y}$ is the same as in (9) with $\rho = 0$.
- Therefore we can use the formulas for Marshallian demands for the CES -case to get:

$$x(p_x, p_y, w) = \frac{\alpha w}{p_x}, \qquad y(p_x, p_y, w) = \frac{(1-\alpha)w}{p_y}.$$

UMP: Cobb-Douglas utility function

- For the Cobb-Douglas utility function, you get the result that the expenditure share $\frac{p_x x}{w} = \alpha$ and $\frac{p_y y}{w} = 1 - \alpha$.
- This extends easily to the case with n goods and $u(x) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha_i > 0, \sum_i \alpha_i = 1$ at prices $p = (p_1, ..., p_n)$. Then you have:

$$x_i(p, w) = \frac{\alpha_i w}{p_i}.$$

- Expenditure shares do not depend on prices and on income.
- In equation (9), you can see that for general CES -functions, expenditure shares depend on prices, but not on income.
- This is not very realistic

UMP: Stone-Geary utility function

- One way to get more realistic consumption patters is to define the utility function for consumptions above a level needed for subsistence.
- Let $\underline{x} = (\underline{x_1}, ..., \underline{x_n})$ be the levels of each good needed for survival and assume that $w \geq p \cdot \underline{x}$.
- The utility function for $x \in \mathbb{R}^n$ such that $x_i \geq \underline{x_i}$ is of Cobb-Douglas -like form:

$$u(x) = (x_1 - \underline{x_1})^{\alpha_1} \dots (x_n - \underline{x_n})^{\alpha_n},$$

  where $0 < \alpha_i < 1$ for all i and $\sum_{i=1}^{n} \alpha_i = 1$.
- Notice that the marginal utility for good i is infinite if $x_i = \underline{x_i}$ and that the utility function is strictly increasing in all of its components.
- Hence we still have an interior solution and the budget constraint binds.

UMP: Stone-Geary utility function

- We get as above:

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_k}} = \frac{\alpha_i(x_k - \underline{x_k})}{\alpha_k(x_i - \underline{x_i})} = \frac{p_i}{p_k} \text{ for all } i, k,$$

$$\sum_{i=1}^{n} p_i x_i = w.$$

- Taking $k = 1$, we get that

$$x_i - \underline{x_i} = \frac{\alpha_i p_1}{\alpha_1 p_i}(x_1 - \underline{x_1}) \text{ for all } i. \tag{10}$$

- Multiplying both sides by p_i and summing over i gives:

$$\sum_{i=1}^{n} p_i(x_i - \underline{x_i}) = \frac{p_1 \sum_{i=1}^{n} \alpha_i}{\alpha_1}(x_1 - \underline{x_1}).$$

UMP: Stone-Geary utility function

- So we can solve:

$$x_1 - \underline{x_1} = \frac{\alpha_1 (w - \sum_{i=1}^n p_i \underline{x_i})}{p_1},$$

where we used the budget constraint $\sum_{i=1}^n p_i x_i = w$ and $\sum_{i=1}^n \alpha_i = 1$

- By (10), we see that

$$x_i - \underline{x_i} = \frac{\alpha_i (w - \sum_{j=1}^n p_j \underline{x_j})}{p_i}.$$

- The consumer uses a constant fraction of her excess income (above what is needed for the necessities $\underline{x}$) in constant shares given by the α_i.
- Since the poor have less excess wealth, their consumption fractions are closer to the ones given by the subsistence levels $\beta_i := \frac{\underline{x_i}}{\sum_i \underline{x_i}}$.

Quasilinear utility function

- We end the section on utility maximization with $u(x, y) = v(x) + y$, where v is a strictly increasing and strictly concave function subject to non-negativity of x, y and the budget constraint $p_x x + y \leq w$.
- Now $MRS_{x,y} = v'(x)$.
- If $v'(\frac{w}{p_x}) > p_x$, then we have a corner solution $x(p_x, w) = \frac{w}{p_x}, y(p_x, w) = 0$. Otherwise $x(p_x, w)$ solves

$$v'(x) = p_x,$$

  and

$$y = (w - p_x x(p_x, w)).$$

- This utility function lies behind partial equilibrium analysis in microeconomics where x is sold in the market of interest and y is everything else.

Expenditure minimization problem

- We cover briefly the related problem of minimizing expenditure subject to the constraint of reaching a specified level of utility.
- All the notation is exactly as in UMP and we assume that the utility function $u(x)$ is quasiconcave.

$$\min_{x \in X} p \cdot x = \sum_{i=1}^{n} p_i x_i,$$

subject to

$$u(x) \geq \overline{u}.$$

- This means that we have a linear and thus quasiconvex objective function for our minimization problem and since the utility function is quasiconcave, the feasible set is convex.

Expenditure minimization problem

- Hence we know that K-T necessary conditions are also sufficient. Notice that the feasible set is now not bounded (why?)
- A solution exists because we can take any x^* such that $u(x^*) \geq \overline{u}$ and restrict attention to x such that $p \cdot x \leq p \cdot x^*$ since x^* is a feasible solution.
- But this set is convex and bounded since it is a budget set.

Expenditure minimization problem

- The Lagrangean to the problem is:

$$\mathcal{L}(x, \lambda) = \sum_{i=1}^{n} p_i x_i - \lambda_0(\overline{u} - u(x)) + \sum_{i=1}^{n} \lambda_i x_i.$$

- The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = p_i + \lambda_0 \frac{\partial u}{\partial x_i} + \lambda_i = 0 \text{ for all } i, \tag{11}$$

$$\lambda_0[u(x) - \overline{u}] = 0, \tag{12}$$

$$\lambda_i x_i = 0 \text{ for all } i, \tag{13}$$

$$\overline{u} - u(x) \leq 0, \tag{14}$$

$$-x_i \leq 0 \text{ for all } i, \tag{15}$$

$$\lambda_i \geq 0 \quad i \in \{0, 1, ..., n\}. \tag{16}$$

Expenditure minimization problem

- Notice that for interior solutions (where $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$, we get again (after eliminating the multiplier) from the first line of the K-T conditions that

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_k}} = \frac{p_i}{p_k}.$$

- The solutions $x_i(p, \overline{u})$ to this minimization problem are called the Hicksian or compensated demands.
- We have exactly the same situation as before. Now the ratio of marginal utilities is really the MRT for the problem since it describes the feasible set.
- The price ratio is now the MRS of this new problem. We will relate these two problems in the next lecture.

Cost minimization problem for a firm

- A firm chooses its inputs k, l to minimize the cost of reaching a production target of $\overline{q}$ at given input prices r, w.
- The production function is assumed to be a strictly increasing and quasiconcave function $f(k, l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk + wl$$

subject to

$$f(k, l) \geq \overline{q}.$$

- Notice that this is the same mathematical problem as in expenditure minimization. Only the names of variables have changed. The solution to the problem is therefore also identical.