

Mathematics for Economists: Lecture 10

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Spring 2020

This lecture covers

1. The value function
2. Interpreting the multipliers
3. Duality in consumer choice
4. Value functions for profit maximizing firms

The value function: motivating example

- ▶ Let's begin with a concrete example.
- ▶ A profit maximizing monopolist firm selling in a market where the market demand curve is given by $p = a - bq$ with $a, b > 0$.
- ▶ The cost of producing q units is cq with $0 < c < a$.
- ▶ The profit is equal to revenue net of cost, i.e. the solves

$$\max_{q \geq 0} (a - bq)q - cq.$$

The value function

- ▶ Since profit at $q = 0$ is zero, the inequality constraint is not binding.
- ▶ Since the objective function is strictly concave, any point satisfying the first-order condition is an optimum.
- ▶ Setting the derivative of the objective function with respect to q at zero gives:

$$q^* = \frac{a - c}{2b}.$$

- ▶ The maximum profit that the firm can get is therefore

$$\pi(a, b, c) = (a - c)q^* - b(q^*)^2 = \frac{(a - c)^2}{4b}.$$

The value function

- ▶ How does the optimal profit depend on the parameters a, b, c ? Since the optimal q changes, maybe this is quite complicated?
- ▶ Just take the derivatives of the function $\pi(a, b, c)$ with respect to its variables.
- ▶ We see that $\frac{\partial \pi(a, b, c)}{\partial a} = \frac{a-c}{2b} = -\frac{\partial \pi(a, b, c)}{\partial c}$, and $\frac{\partial \pi(a, b, c)}{\partial b} = -\frac{(a-c)^2}{4b^2}$.
- ▶ But it is also true that $\frac{\partial \pi(a, b, c)}{\partial a} = q^* = \frac{\partial \pi(a, b, c)}{\partial a}$ and $\frac{\partial \pi(a, b, c)}{\partial c} = -(q^*)^2$.
- ▶ But these are the partial derivatives of the objective function with respect to each parameter
- ▶ Is there a reason behind this or is this just a coincidence?

The value function

- ▶ Consider an unconstrained maximization problem of a function of a single real variable x , where the objective function depends on a parameter $\alpha \in \mathbb{R}$.

$$\max_{x \in \mathbb{R}} f(x, \alpha).$$

- ▶ Let $x(\alpha)$ be the solution to this problem.
- ▶ Consider the maximum value of the objective function that is achievable at the exogenous variable (or parameter) $\hat{\alpha}$, i.e. $f(x(\hat{\alpha}), \hat{\alpha})$.
- ▶ We call this new function the *value function* of the problem and denote

$$V(\alpha) := f(x(\alpha), \alpha).$$

The value function

- ▶ At the (unconstrained) optimum $x(\hat{\alpha})$, by the first-order condition:

$$\frac{\partial f(x(\hat{\alpha}), \alpha)}{\partial x} = 0.$$

Suppose that f is twice continuously differentiable and that the second order condition is satisfied so that

$$\frac{\partial^2 f(x(\hat{\alpha}))}{\partial x^2} < 0.$$

- ▶ Then we can use implicit function theorem to see that $x(\alpha)$ satisfying the first-order condition $x'(\alpha) = 0$ exists in some neighborhood of $\hat{\alpha}$.

The value function

- ▶ By chain rule:

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} x'(\hat{\alpha}) + \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

- ▶ Since $\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} = 0$, we get

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

The value function: envelope theorem

- ▶ The envelope theorem states that for twice continuously differentiable functions $f(x, \alpha)$, and the value function $V(\alpha) = \max_x f(x, \alpha)$, we have

$$V'(\alpha) = \frac{\partial f(x(\alpha), \alpha)}{\partial \alpha}.$$

- ▶ In words, when a parameter changes, the maximum value of the problem changes only through the direct effects on the objective function.
- ▶ Indirect effects on the value vanish because of the first-order condition on x .
- ▶ Can you relate the theorem to the motivating example?

The value function

- ▶ In the more general case, where $x \in \mathbb{R}^n$, the message is exactly the same. The first order-condition is now:

$$\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = 0 \text{ for all } i \in \{1, \dots, n\}.$$

- ▶ Assuming the conditions for implicit function theorem, we have by chain rule:

$$V'(\hat{\alpha}) = \sum_{i=1}^n \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x'_i(\hat{\alpha}) + \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

- ▶ Again, the first term vanishes by first-order condition and we are left with

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Envelope theorem in equality constrained problems

- ▶ Suppose that we have an equality constrained parametric maximization problem for $x \in \mathbb{R}^n$:

$$\begin{aligned} & \max_x f(x, \alpha) \\ & \text{subject to } g(x, \alpha) = 0. \end{aligned}$$

- ▶ Let $x(\alpha)$ denote the optimal solution and assume sufficient differentiability that we can use implicit function theorem around the solution as before.
- ▶ The value function is still defined as: $V(\alpha) = f(x(\alpha), \alpha)$.
- ▶ Form the Lagrangean:

$$\mathcal{L}(x, \mu; \alpha) = f(x, \alpha) - \mu g(x, \alpha).$$

Envelope theorem in equality constrained problems

Theorem (Envelope theorem)

In an optimization problem subject to an equality constraint, we have:

$$V'(\alpha) = \frac{\partial \mathcal{L}(x, \mu; \alpha)}{\partial \alpha}.$$

Proof.

$$V'(\hat{\alpha}) = \sum_{i=1}^n \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x'_i(\hat{\alpha}) + \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Now the first-order condition implies that

$$\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = \mu \frac{\partial g(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i}.$$

Envelope theorem in equality constrained problems

Since the constraint holds for all α , we have

$$\sum_{i=1}^n \frac{\partial g(x(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x'_i(\hat{\alpha}) = - \frac{\partial g(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Combining these gives:

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha} - \mu \frac{\partial g(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$



Interpreting the Lagrange multipliers

- ▶ Envelope theorem gives us a nice way of understanding the Lagrange multipliers in UMP
- ▶ The Lagrangean for the UMP with a single binding equality constraint is:

$$\mathcal{L}(x, \lambda) = u(x) - \mu \left[\sum_{i=1}^n p_i x_i - w \right].$$

- ▶ The maximum value function

$$v(p, w) = \max u(x) \text{ subject to } p \cdot x = w,$$

is called the indirect utility function. It computes the optimal utility level for all combinations of prices $p \in \mathbb{R}_{++}^n$ and income $w > 0$.

Interpreting the Lagrange multipliers

- ▶ Envelope theorem tells us that

$$\frac{\partial v(p, w)}{\partial w} = \mu.$$

- ▶ This means that if your income is increased by one unit, your maximal utility increases the amount given by the multiplier.
- ▶ By reducing income dw you lose μdw of utility and this is why the multiplier is sometimes called the shadow price of income.
- ▶ Note also that the usual first-order condition requires:

$$\mu dw = \frac{\partial u(x)}{\partial x_i} \frac{dw}{p_i} \text{ for all } i.$$

- ▶ With dw of additional income, you can buy $\frac{dw}{p_i}$ units of good i .

Interpreting the Lagrange multipliers

- ▶ Envelope theorem also tells us that:

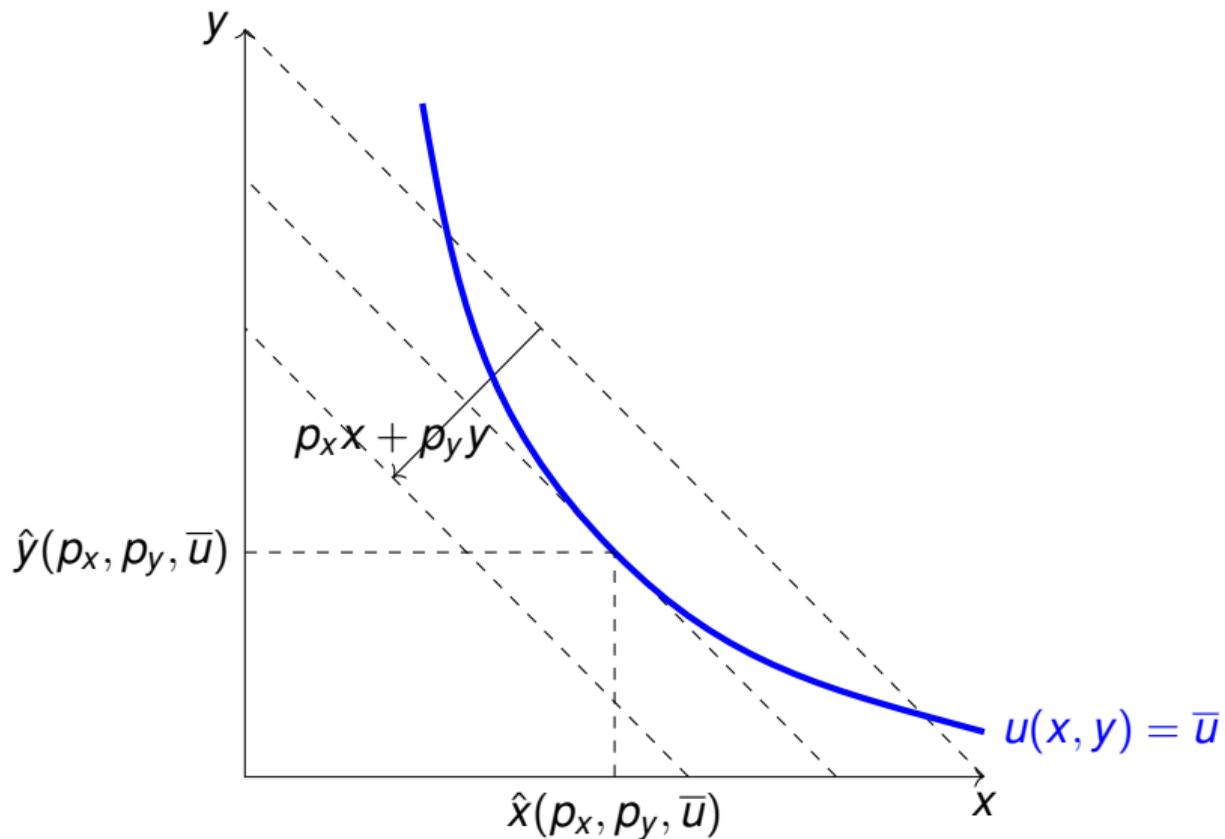
$$\frac{\partial v(p, w)}{\partial p_i} = -\mu x_i.$$

- ▶ Combining these two, we have Roy's identity:

$$x_i(p, w) = -\frac{\frac{\partial v(p, w)}{\partial p_i}}{\frac{\partial v(p, w)}{\partial w}}.$$

- ▶ In other words, if you have an indirect utility function, you can compute the demand function by simple partial differentiation.
- ▶ In later courses, you will learn what properties on $v(p, w)$ guarantee that it is the indirect utility function of some UMP for some $u(x)$.

Figure: Expenditure minimization



Expenditure minimization

- ▶ Consider next the expenditure minimization problem from Lecture 9.

$$\min_{h \in X} p \cdot x = \sum_{i=1}^n p_i h_i,$$

subject to

$$u(h) = \bar{u}.$$

- ▶ Denote the solution to this problem by $h(p, u)$. We call $h_i(p, \bar{u})$ the Hicksian or compensated demand for good i .
- ▶ The (minimum) value function of this problem $e(p, \bar{u}) = \sum_{i=1}^n p_i h_i(p, \bar{u})$ is called the expenditure function.
- ▶ The objective function is linear in p and hence by the results in Lecture 6, we know that $e(p, \bar{u})$ is concave in p .
- ▶ Therefore the Hessian matrix of $e(p, \bar{u})$ is negative semidefinite.

Value function for expenditure minimization

- ▶ The Lagrangean for interior solutions:

$$\mathcal{L}(x, \mu) = \sum_{i=1}^n p_i h_i - \mu(\bar{u} - u(h)).$$

- ▶ Envelope theorem tells us that

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i.$$

- ▶ The partial derivatives of $h_i(p, \bar{u})$ with respect to p_j are the elements of the Hessian matrix of $e(p, \bar{u})$.

Connecting expenditure minimization and UMP

- ▶ Hold prices \hat{p} fixed for a moment and ask how high utility you can achieve with income w . The answer is given by the indirect utility function $v(\hat{p}, w)$.
- ▶ Ask next what is the minimum expenditure that you must use to achieve utility $v(\hat{p}, w)$. The following figures should convince you that for all \hat{p} ,

$$e(\hat{p}, v(\hat{p}, w)) = w.$$

- ▶ It costs you $e(\hat{p}, \bar{u})$ to reach utility \bar{u} . If your budget is $e(\hat{p}, \bar{u})$, then the maximal utility that you can reach is for all \hat{p} ,

$$\bar{u} = v(\hat{p}, e(\hat{p}, \bar{u})).$$

Figure: UMP for $w = e(p, v(p, w))$

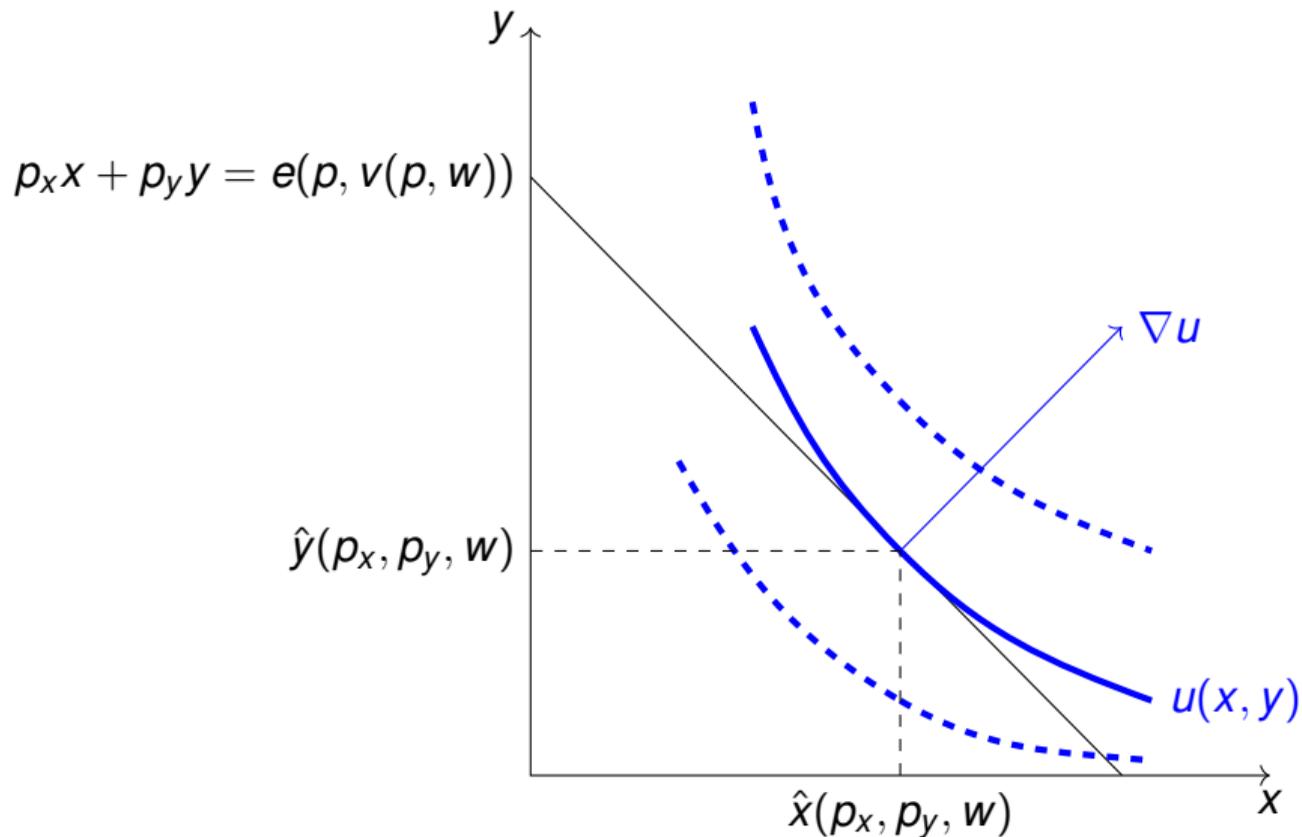
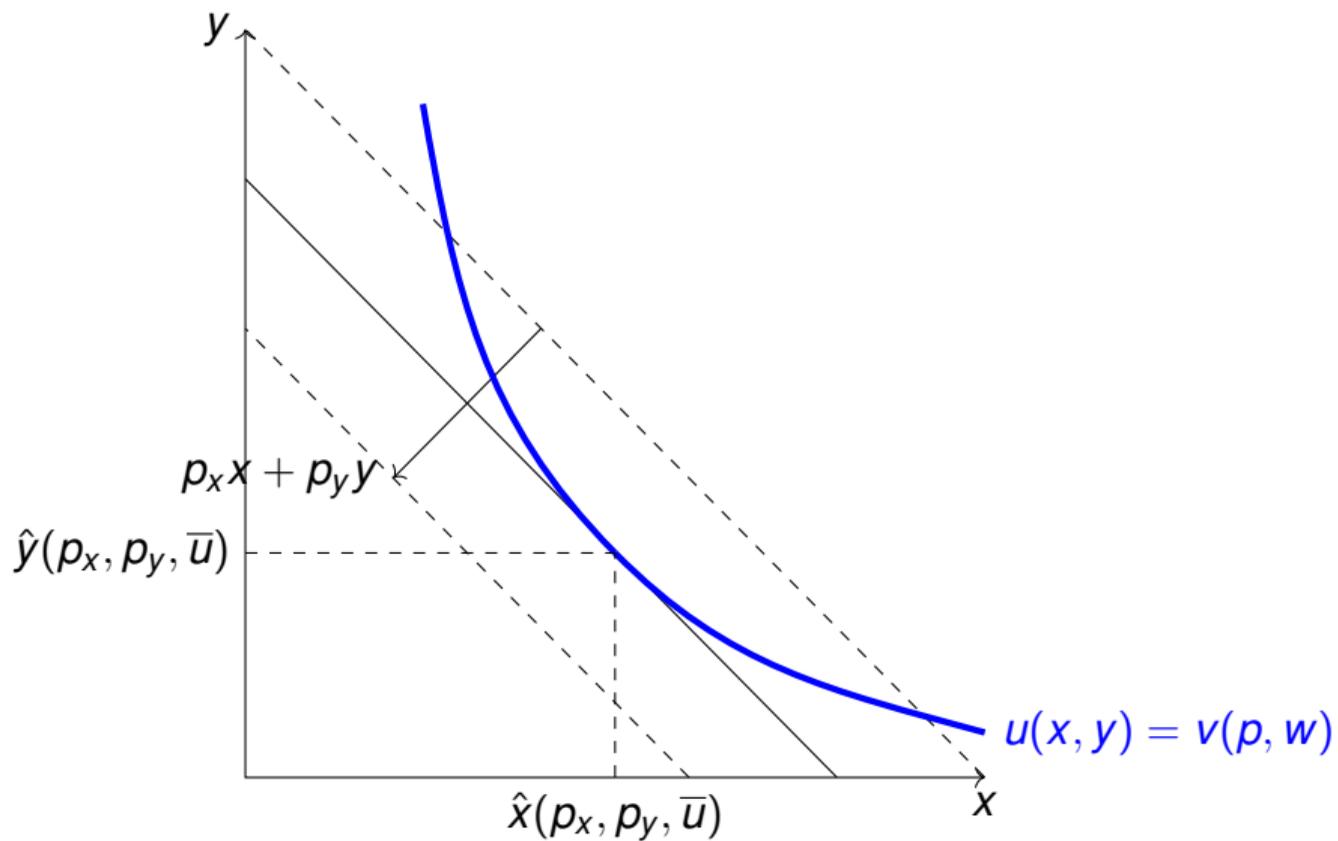


Figure: Expenditure minimization for $\bar{u} = v(p, w)$



Connecting expenditure minimization and UMP

- ▶ You can also see that for $\bar{u} = v(p, e(p, \bar{u}))$ and $e(p, v(p, w)) = w$ the solutions to expenditure minimization and UMP coincide for all p :

$$h_i(p, \bar{u}) = x_i(p, e(p, \bar{u})) \text{ for all } i,$$

$$h_i(p, v(p, w)) = x_i(p, w) \text{ for all } i.$$

Connecting expenditure minimization and UMP

- ▶ Differentiate the first of these identities with respect to p_j to get:

$$\begin{aligned}\frac{\partial h_i(p, \bar{u})}{\partial p_j} &= \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} \frac{\partial e(p, \bar{u})}{\partial p_j} \\ &= \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} h_j(p, \bar{u}) \\ &= \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, e(p, \bar{u})) \\ &= \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w).\end{aligned}$$

Connecting expenditure minimization and UMP

- ▶ This is the famous Slutsky equation for income and substitution effects.
- ▶ The observable change in Marshallian $\frac{\partial x_i(p, w)}{\partial p_j}$ demands can be decomposed into a substitution effect, i.e. the change in compensated demand $\frac{\partial h_i(p, \bar{u})}{\partial p_j}$ and the observable income effect $\frac{\partial x_i(p, w)}{\partial w} x_j(p, w)$.

$$\frac{\partial x_i(p, w)}{\partial p_j} = \frac{\partial h_i(p, \bar{u})}{\partial p_j} - \frac{\partial x_i(p, w)}{\partial w} x_j(p, w).$$

- ▶ Since we know that the Hessian of $e(p, \bar{u})$ is negative definite, we know that its diagonal elements are non-positive.
- ▶ Hence the effect of increasing p_j on x_j is negative whenever the demand for j is increasing in income (we say then that j is a non-inferior good).

Cost function of a competitive firm

- ▶ A firm chooses its inputs k, l to minimize the cost of reaching a production target of \bar{q} at given input prices r, w .
- ▶ The production function is assumed to be a strictly increasing and quasiconcave function $f(k, l)$.

$$\min_{(k,l) \in \mathbb{R}_+^2} rk + wl$$

subject to

$$f(k, l) = q.$$

Cost function of a competitive firm

- ▶ The value function of this problem is called the *cost function* of the firm and denoted by $c(r, w, q)$.

$$c(r, w, q) = rk(r, w, q) + wl(r, w, q),$$

where $k(r, w, q), l(r, w, q)$ solve the cost minimization problem.

- ▶ These are called the conditional factor demands.
As in the case with expenditure minimization, we see that the cost function is concave in r, w since it is the minimum of linear functions of r, w .
- ▶ Therefore the Hessian of the cost function is negative semidefinite.

Cost function of a competitive firm

- ▶ By envelope theorem, we have the result known as Shephard's lemma:

$$\frac{\partial c(r, w, q)}{\partial r} = k(r, w, q), \quad \frac{\partial c(r, w, q)}{\partial w} = l(r, w, q).$$

- ▶ Negative semidefiniteness of the Hessian of c implies that (since the diagonal elements must be non-positive)

$$\frac{\partial k(r, w, q)}{\partial r} \leq 0, \quad \frac{\partial l(r, w, q)}{\partial w} \leq 0.$$

- ▶ In words, conditional factor demands are decreasing in own price (not surprisingly).

Profit function of a competitive firm

- ▶ We end this part of the course with the analysis of profit maximization for a price taking firm.
- ▶ There are two ways to think about this. Either minimize cost for each production level q to get $c(r, w, q)$ and then choose q optimally to maximize $pq - c(r, w, q)$, where p is the price of the output.
- ▶ Alternatively, you can write directly:

$$\max_{k,l,q} pq - rk - wl$$

subject to

$$q = f(k, l).$$

Profit function of a competitive firm

- ▶ An advantage of the second approach is that the problem is immediately seen to be linear in the input and output prices p, r, w .
- ▶ Let $q(p, r, w), k(p, r, w), l(p, r, w)$ be the optimal output and input choices in the problem. The value function $\pi(p, r, w)$ is called the profit function of the firm.
- ▶ Since π is the maximum of linear functions in p, r, w , we get by Lecture 6 that π is convex and hence its Hessian is positive semidefinite.

Profit function of a competitive firm

- ▶ The envelope theorem gives us Hotelling's lemma:

$$\frac{\partial \pi(p, r, w)}{\partial p} = q(p, r, w), \quad \frac{\partial \pi(p, r, w)}{\partial r} = -k(p, r, w), \quad \frac{\partial \pi(p, r, w)}{\partial w} = -l(p, r, w).$$

- ▶ Since π is positive semi-definite, its diagonal elements are non-negative.
- ▶ This gives the 'Law of Supply' (supply increases in output price)

$$\frac{\partial q(p, r, w)}{\partial p} \geq 0,$$

and the 'Law of Factor Demands' (factor demand decrease in factor price):

$$\frac{\partial k(p, r, w)}{\partial r} \leq 0, \quad \frac{\partial l(p, r, w)}{\partial w} \leq 0.$$

Profit function of a competitive firm

- ▶ As you can see, the theory of the competitive firm is easier than consumer theory since changes in prices do not change the constraint set (as with the budget set)
- ▶ You will see the firm's problem in some form in almost all branches of economics and in particular in Intermediate Microeconomics
- ▶ Of course, in many industries firms are not competitive → Industrial organization
- ▶ Firms do not make decisions but people do and people may have different objectives → Organizational economics, Contract theory