Mathematics for Economists

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Solutions to the problem set 4:

Question 1:

$$\max f(x,y) = xy$$

$$subject\ to: \quad x + y \leq 100$$

$$x \leq 40$$

$$x, y \geq 0$$

We first form the Lagrangian:

$$L = xy - \lambda_1(x + y - 100) - \lambda_2(x - 40) + \lambda_3 x + \lambda_4 y = 0$$

The first order conditions:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow y - \lambda_1 - \lambda_2 + \lambda_3 = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x - \lambda_1 + \lambda_4 = 0 \quad (2)$$

$$\lambda_1(x + y - 100) = 0 \quad (3)$$

$$\lambda_2(x - 40) = 0 \quad (4)$$

$$\lambda_3 x = 0\,, \lambda_4 y = 0$$

$$x + y \leq 100, \qquad x \leq 40, \qquad x, y \geq 0$$

If $x, y > 0$, then $\lambda_3, \lambda_4 = 0$. Putting this into (1) and (2) we have:

$$x = \lambda_1$$

$$y - x = \lambda_2$$

Using (3) we have:

$$x(x + y - 100) = 0$$

And $x > 0$, so first condition is binding and:

$$x + y = 100 \qquad (5)$$

Using (4)

$$(y - x)(x - 40) = 0 \Rightarrow \begin{cases} y = x \quad not\ valid\ because\ of\ (5)\ and\ the\ fact\ that\ x \leq 40 \\ x = 40\ valid, so\ the\ second\ condition\ is\ binding \end{cases}$$

So $x^* = 40$. And using (5) we have

$$y^* = 100 - 40 = 60$$

Question 2:

$$x \in R_+^n\ , p_i > 0\ , w > 0\ and$$

$$p.x \le w$$

$$\sum_{i=1}^{n} p_i x_i \le w$$

Since $p_i's\ and\ x_i's$ are both positive, the maximum value of each x_i is when we set all the other elements, $x_j\ for\ j \neq i$ equal to 0, so:

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n = w\ , \qquad p_i, x_i > 0$$

$$\Rightarrow \bar{x}_i = \max(x_i) = \frac{w}{p_i} \quad when\ x_j = 0\ for\ all\ j \neq i$$

Now what we are looking for is a bound for $||x|| = \sum_{i=1}^{n} x_i^2$, but we know the upper limit for each x_i so:

$$\forall\ x_i : x_i \le \frac{w}{p_i} \Rightarrow \sum_{i}^{n} x_i^2 \le \sum_{i}^{n} \frac{w^2}{p_i^2} \Rightarrow ||x|| \le \frac{w}{||p||}$$

So the set B is bounded.

Extra credit:

To prove that a set is closed we should prove that every limit point of the set is in the set. We name the set of $x \in R_+^n$ for which we have $p.x \le w$ to be A, so:

$$x \in A: \quad p.x \le w$$

Now assume that $\hat{x}$ is a limit point of the set A, then there is a sequence $\{x_n\}_{n\ge0} \subset A$ such that: (Lecture 7, page 13, theorem iv)

$$\lim_{n\to\infty} x_n = \hat{x}$$

$$\text{As } \{x_n\}_{n\ge0} \subset A \Rightarrow p.x_n \le w, \forall n$$

This gives:

$$\lim_{n\to\infty} p.x_n \le w \Rightarrow \quad p.\hat{x} \le w$$

So $\hat{x}$ is in the set A, so the set A is closed.

Question 3:

a)

$$\max_{(x_1,\dots,x_n)\in R_+^n} (x_1.x_2.\dots x_n)^{\frac{1}{n}}$$

subject to $\sum_{i=1}^{n} x_i = y$

i)

We form the Lagrangian first:

$$L(x_1, \dots, x_n, \lambda) = (x_1 . x_2 . \dots \; x_n)^{\frac{1}{n}} + \lambda(\sum_{i=1}^{n} x_i - y)$$

The objective function is obviously continuous. Moreover, all the x_is are positive and sum of them is equal to y, so: $0 \le x_i \le y$, so the feasible set is compact and the optimization have solutions. Then we take derivatives of the Lagrangian to get the FOCs.

ii)

$$\frac{\partial L}{\partial x_1} = (x_2 \dots x_n)^{\frac{1}{n}} . \frac{1}{n} x_1^{\frac{1-n}{n}} - \lambda = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = (x_1 \dots x_{n-1})^{\frac{1}{n}} . \frac{1}{n} x_n^{\frac{1-n}{n}} - \lambda = 0$$

$$\sum_{i=1}^{n} x_i - y = 0$$

iii, iv)

from the first and the second foc:

$$(x_2 \dots x_n)^{\frac{1}{n}} . \frac{1}{n} x_1^{\frac{1-n}{n}} = (x_1, x_3 \dots x_n)^{\frac{1}{n}} . \frac{1}{n} x_2^{\frac{1-n}{n}}$$

$$x_1^{\frac{1-n}{n}} x_2^{\frac{1}{n}} = x_2^{\frac{1-n}{n}} x_1^{\frac{1}{n}} \Rightarrow x_1 = x_2$$

Using the rest of the constraints we get

$$x_1 = x_2 = \cdots = x_n$$

And finally using the last constraint we get

$$x_1^* = x_2^* = \cdots = x_n^* = \frac{y}{n}$$

And this is the unique solution of the problem, and in this optimal solution the two averages are equal to each other.

Arithmetic mean=$\frac{1}{n} \sum_{i=1}^{n} x_i = \frac{y}{n}$

Geometric mean=$(\Pi_{i=1}^{n} x_i)^{\frac{1}{n}} = \frac{y}{n}$

b)

$$\max_{(x_1, \dots, x_n) \in R_+^n} \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

$subject\ to$ $\sum_{i=1}^{n} x_i = y$

We first form the Langrangian. So:

$$L = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} + \lambda(\sum_{i=1}^{n} x_i - y)$$

$$\frac{\partial L}{\partial x_1} = n\left(\frac{\frac{1}{x_1^2}}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}\right) + \lambda = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = n\left(\frac{\frac{1}{x_n^2}}{(\sum_{i=1}^{n} \frac{1}{x_i})^2}\right) + \lambda = 0$$

$$\sum_{i=1}^{n} x_i = y$$

Just like before, all the constraints we get:

$$x_1^* = x_2^* = \cdots = x_n^* = \frac{y}{n}$$

And finally at the optimum point we have:

Arithmetic mean=$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{y}{n}$

harmonic mean=$\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} = \frac{y}{n}$

Question 4:

a) For convex functions we have:

$$f(\theta a + (1-\theta)b) \le \theta f(a) + (1-\theta)f(b)$$

And for concave functions:

$$f(\theta a + (1-\theta)b) \ge \theta f(a) + (1-\theta)f(b)$$

So if the function f is convex and concave at the same time then:

$$f(\theta a + (1-\theta)b) = \theta f(a) + (1-\theta)f(b)$$

To prove that the only function that has the above characteristics is the affine function we should prove that:

1- $f(\alpha x) = \alpha f(x)$ and
2- $f(a+b) = f(a) + f(b)$

Without loss of generality we assume that $f(0) = 0$ (if not we can easily assume function g, where $g(x) = f(x) - f(0)$ and we need the lemma that if function f is both concave and convex then the function $f(x) - f(0)$ is both convex and concave and then prove the characteristics for function g.)

1- $f(\alpha x) = f(\alpha x + (1-\alpha) * 0) = \alpha f(x) + (1-\alpha)f(0) = \alpha f(x)$
2- $f(a+b) = f\left(\frac{1}{2}.2a + \frac{1}{2}.2b\right) = \frac{1}{2}f(2a) + \frac{1}{2}f(2b) = f(a) + f(b)$

So f is an affine function.

b) According to the definitions of the quasiconcave and quasiconvex functions:

Quasi concave functions: $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$

Quasi convex functions: $f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$

So any increasing or decreasing function f in R_+ is both quasiconcave and quasiconves, such as:

$$f(x) = x^2 \quad or \quad f(x) = e^x \quad \ldots$$

c) Assume the point to be:

$$p_0 = (x_0, y_0, z_0)$$

And the plane N to be:

$$N: Ax + By + Cz + D = 0$$

So the optimization problem is:

$$\min_{(x,y,z)\in N} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$
$$subject\ to \qquad Ax + By + Cz + D = 0$$

We can equivalently write it as:

$$\max_{(x,y,z)\in N} -[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]$$
$$subject\ to \qquad Ax + By + Cz + D = 0$$

Now the objective function is a concave function and the constraint is the set of all the points on a plain which is obviously a convex set.

d)

$f(x)$ is a concave function so: $f''(x) \leq 0$

$$g(y)\ is\ increasing\ and\ concave\ so\ \ g'(x) \geq 0\ and\ \ g''(x) \leq 0$$

$$\frac{d}{dx} g(f(x)) = f'(x) g'(f(x))$$

$$\frac{d^2}{d^2x} g(f(x)) = f''(x) g'(f(x)) + (f'(x))^2 g''(f(x))$$

Form the assumptions that $f(x)$ is a concave and $g(y)$ is increasing and concave we have:

$f''(x) g'(f(x)) \leq 0$ and

$$(f'(x))^2 g''(f(x)) \leq 0$$

So after all

$$\frac{d^2}{d^2x} g(f(x)) \leq 0$$

And the $g(f(x))$ is also concave.

Question 5:

a)

We form the Hessian matrix of the function u:

$$u(f,c,s) = \alpha_f \ln(f) + \alpha_c \ln(c) + \alpha_s \ln(s)$$

So

$$H_u = \begin{bmatrix} \frac{\partial^2 u}{\partial f^2} & \frac{\partial^2 u}{\partial f \partial c} & \frac{\partial^2 u}{\partial f \partial s} \\ \frac{\partial^2 u}{\partial c \partial f} & \frac{\partial^2 u}{\partial c^2} & \frac{\partial^2 u}{\partial c \partial s} \\ \frac{\partial^2 u}{\partial f \partial s} & \frac{\partial^2 u}{\partial s \partial c} & \frac{\partial^2 u}{\partial s^2} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_f}{f^2} & 0 & 0 \\ 0 & -\frac{\alpha_c}{c^2} & 0 \\ 0 & 0 & -\frac{\alpha_s}{s^2} \end{bmatrix}$$

Which is obviously negative definite for all the (f,c,s) in the domain so the function u is strictly concave.

b)

So the time of the father is allocated to three different tasks:

$$s + c + h_w = 24$$

Where h_w is the working time. Moreover the income should be equal to the expenses so:

$$h_w . w \geq f$$

So over all

$$(24 - s - c).w - f \geq 0$$

c)

The feasible set is:

$$g(f,c,s) = (24 - s - c).w - f \geq 0$$

$$0 < s \leq 24$$

$$0 < c \leq 24$$

$$0 < f \leq 24w$$

d)

$$\max_{(f,c,s) \in R_+^3} \alpha_f \ln(f) + \alpha_c \ln(c) + \alpha_s \ln(s)$$

Subject to:

$$g(f,c,s) = (24 - s - c).w - f \geq 0$$

$$s, c, f > 0$$

Now we form the Lagrangian:

$$L = \alpha_f \ln(f) + \alpha_c \ln(c) + \alpha_s \ln(s) + \lambda((24 - s - c).w - f)$$

Then the first order conditions:

$$\frac{\partial L}{\partial f} = \frac{\alpha_f}{f} - \lambda = 0$$

$$\frac{\partial L}{\partial c} = \frac{\alpha_c}{c} - \lambda w = 0$$

$$\frac{\partial L}{\partial s} = \frac{\alpha_s}{s} - \lambda w = 0$$

$$\lambda[(24 - s - c).w - f] = 0$$

e,f) Obviously none of the non-negativity constraints are binding because if they do then the function u will be infinite ($\lim_{x \to 0} \ln(x) = -\infty$)

Using the first three focs:

$$c = \frac{f\alpha_c}{w\alpha_f}$$

$$s = \frac{f\alpha_s}{w\alpha_f}$$

$$and\ \ \lambda \neq 0\ so\ the\ budget\ constraint\ is\ binding$$

$$\left(24 - \frac{f\alpha_s}{w\alpha_f} - \frac{f\alpha_c}{w\alpha_f}\right)w - f = 0$$

$$\frac{24w\alpha_f - f\alpha_s - f\alpha_c}{\alpha_f} - f = 0$$

$$24w\alpha_f = f(\alpha_s + \alpha_c + \alpha_f)$$

$$f^* = \frac{24w\alpha_f}{\alpha_s + \alpha_c + \alpha_f}$$

$$c^* = \frac{24\alpha_c}{\alpha_s + \alpha_c + \alpha_f}$$

$$s^* = \frac{24\alpha_s}{\alpha_s + \alpha_c + \alpha_f}$$

The Weierstrass' theorem is not satisfied because the set E of the feasible solutions is not compact, because it is not closed.

We try to answer this question for the variable s, but it is the same for other variables f and c. We know from the form of the utility function that α_s shows the importance of the sleeping in the father's utility function, and we know that it is always positive so assume that $\alpha_s = \delta$, which is a very small amount. So we have:

$$s^* = \frac{24\delta}{\alpha_s + \alpha_f}$$

And we know that utility function u is not continuous at $(f = 0, c = 0, s = 0)$, BUT it is always possible to consider small enough ε such that:

$$s = \varepsilon < \frac{24\delta}{\alpha_s + \alpha_f}$$

And the utility function will always be continuous at this point, so we will always have an interior solution to our problem no matter how small $\alpha_s, \alpha_c, \alpha_f$ be.