

Lecture 5: Stochastic Runge–Kutta Methods

Arno Solin

IndoorAtlas Ltd. /
Aalto University, Finland

Contents

- 1 Introduction
- 2 Runge–Kutta methods for ODEs
- 3 Strong stochastic Runge–Kutta methods
- 4 Weak stochastic Runge–Kutta methods
- 5 Summary

Overview of this lecture

- **Runge–Kutta** methods for ODEs
 - Taylor series.
 - General Runge–Kutta schemes.
 - Explicit and implicit schemes.
- **Strong** stochastic Runge–Kutta methods
 - Itô–Taylor series.
 - A family of strong order 1.0 schemes.
 - The iterated Itô integrals.
- **Weak** stochastic Runge–Kutta methods
 - A family of weak order 2.0 schemes.
 - Approximating the iterated Itô integrals.

Runge–Kutta: Basic principles

- A family of **iterative methods** for solving differential equations.
- Based on **Taylor series** (see the previous lecture), ...
- ... but are **derivative-free**.
- **Plug-and-play** methods that only require specification of the differential equation (at least ideally).
- There are other methods as well (not considered here):
 - Multistep methods (*e.g.* Adams methods)
 - Multiderivative methods
 - Higher-order methods (*e.g.* Nyström method)
 - Tailored methods (for specific problems)

Runge–Kutta: Motivation

- Consider a first-order non-linear ODE

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(t_0) = \text{given},$$

- The simplest Runge–Kutta method is the (forward) **Euler scheme**.
- It is based on **sequential linearization** of the ODE system:

$$\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_k), t_k) \Delta t.$$

- **Easy** to understand and implement.
- The **global error** of the method depends linearly on the step size Δt .

- The ODE system can be integrated to give

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}(\mathbf{x}(\tau), \tau) d\tau.$$

- Recall from the previous lecture that we used a **Taylor series expansion** for the solution of the ODE

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}(t_0) + \mathbf{f}(\mathbf{x}(t_0), t_0) (t - t_0) \\ &\quad + \frac{1}{2!} \mathcal{L} \mathbf{f}(\mathbf{x}(t_0), t_0) (t - t_0)^2 \\ &\quad + \frac{1}{3!} \mathcal{L}^2 \mathbf{f}(\mathbf{x}(t_0), t_0) (t - t_0)^3 + \dots \end{aligned}$$

- We used the linear operator

$$\mathcal{L}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \sum_i f_i \frac{\partial}{\partial x_i}(\bullet)$$

- In other words, the series expansion is equal to

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(t_0) + \mathbf{f}(\mathbf{x}(t_0), t_0) (t - t_0) \\ &+ \frac{1}{2!} \left\{ \frac{\partial}{\partial t} \mathbf{f}(\mathbf{x}(t_0), t_0) + \sum_i f_i(\mathbf{x}(t_0), t_0) \frac{\partial}{\partial x_i} \mathbf{f}(\mathbf{x}(t_0), t_0) \right\} (t - t_0)^2 \\ &+ \frac{1}{3!} \left\{ \frac{\partial[\mathcal{L} \mathbf{f}(\mathbf{x}(t_0), t_0)]}{\partial t} + \sum_i f_i(\mathbf{x}(t_0), t_0) \frac{\partial[\mathcal{L} \mathbf{f}(\mathbf{x}(t_0), t_0)]}{\partial x_i} \right\} (t - t_0)^3 \\ &+ \dots\end{aligned}$$

- If we were only to consider the **terms up to Δt** , we would recover the **Euler method**.

Derivation of a higher-order method [1/4]

- However, here we wish to get hold of **higher-order methods**.
- For the sake of simplicity, we now stop at the term $(t - t_0)^2 = (\Delta t)^2$.
- We get

$$\mathbf{x}(t_0 + \Delta t) \approx \mathbf{x}(t_0) + \mathbf{f}(\mathbf{x}(t_0), t_0) \Delta t + \frac{1}{2} \left\{ \frac{\partial}{\partial t} \mathbf{f}(\mathbf{x}(t_0), t_0) + \sum_i f_i(\mathbf{x}(t_0), t_0) \frac{\partial}{\partial x_i} \mathbf{f}(\mathbf{x}(t_0), t_0) \right\} (\Delta t)^2.$$

- We aim to get rid of the derivatives and be able to write the expression in terms of the function $\mathbf{f}(\cdot, \cdot)$ evaluated at various points.

Derivation of a higher-order method [2/4]

- We now seek a form with an extra stage:

$$\mathbf{x}(t_0 + \Delta t) \approx \mathbf{x}(t_0) + A\mathbf{f}(\mathbf{x}(t_0), t_0) \Delta t \\ + B\mathbf{f}(\mathbf{x}(t_0) + C\mathbf{f}(\mathbf{x}(t_0), t_0) \Delta t, t_0 + D \Delta t) \Delta t,$$

where A , B , C , and D are unknown.

- In the last term, we can consider the **truncated Taylor expansion** (linearization) around $\mathbf{f}(\mathbf{x}(t_0), t_0)$ with the chosen increments as follows:

$$\mathbf{f}(\mathbf{x}(t_0) + C\mathbf{f}(\mathbf{x}(t_0), t_0) \Delta t, t_0 + D \Delta t) = \mathbf{f}(\mathbf{x}(t_0), t_0) \\ + C \left(\sum_i f_i(\mathbf{x}(t_0), t_0) \frac{\partial}{\partial x_i} \mathbf{f}(\mathbf{x}(t_0), t_0) \right) \Delta t + D \frac{\partial \mathbf{f}(\mathbf{x}(t_0), t_0)}{\partial t} \Delta t + \dots$$

Derivation of a higher-order method [3/4]

- Combining the previous two equations gives:

$$\mathbf{x}(t_0 + \Delta t) \approx \mathbf{x}(t_0) + (A + B) \mathbf{f}(\mathbf{x}(t_0), t_0) \Delta t + B \left[C \sum_i f_i(\mathbf{x}(t_0), t_0) \frac{\partial}{\partial x_i} \mathbf{f}(\mathbf{x}(t_0), t_0) + D \frac{\partial \mathbf{f}(\mathbf{x}(t_0), t_0)}{\partial t} \right] (\Delta t)^2.$$

- If we now compare the above equation to the original truncated Taylor expansion, we get the following conditions for our coefficients:

$$A + B = 1, \quad B = \frac{1}{2}, \quad C = 1, \quad \text{and} \quad D = 1.$$

Derivation of a higher-order method [4/4]

- What we derived here is a **two-stage** method (known as **Heun's method**):

$$\hat{\mathbf{x}}(t_0 + \Delta t) = \mathbf{x}(t_0) + \frac{\Delta t}{2} \{ \mathbf{f}(\tilde{\mathbf{x}}_1, t_0) + \mathbf{f}(\tilde{\mathbf{x}}_2, t_0 + \Delta t) \},$$

where the supporting values are given by

$$\begin{aligned}\tilde{\mathbf{x}}_1 &= \mathbf{x}(t_0), \\ \tilde{\mathbf{x}}_2 &= \mathbf{x}(t_0) + \mathbf{f}(\tilde{\mathbf{x}}_1, t_0) \Delta t.\end{aligned}$$

- The method (in practice the **finite differences**) are determined by the choices we did in truncating the series expansion.
- This method is of order 2.

A general Runge–Kutta method

Algorithm: Runge–Kutta method

Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$ into n steps $t_0 < t_1 < t_2 < \dots < t_n = t$ such that $\Delta t = t_{k+1} - t_k$. The integration method is defined by its **Butcher tableau**:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \boldsymbol{\alpha}^\top \end{array}$$

On each step k approximate the solution as follows:

$$\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \sum_{i=1}^s \alpha_i \mathbf{f}(\tilde{\mathbf{x}}_i, \tilde{t}_i) \Delta t,$$

where $\tilde{t}_i = t_k + c_i \Delta t$ and $\tilde{\mathbf{x}}_i = \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j} \mathbf{f}(\tilde{\mathbf{x}}_j, \tilde{t}_j) \Delta t$.

- Ordinary Runge–Kutta methods are commonly expressed in terms of a table called the **Butcher tableau**:

c_1	$A_{1,1}$			
c_2	$A_{2,1}$	$A_{2,2}$		
\vdots	\vdots		\ddots	
c_s	$A_{s,1}$	$A_{s,2}$	\dots	$A_{s,s}$
	α_1	α_2	\dots	α_s

Example: Forward Euler

Example (Forward Euler)

The forward Euler scheme has the Butcher tableau:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

which gives the recursion $\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_k), t_k) \Delta t$.

Example: RK4

Example (The fourth-order Runge–Kutta method)

The well-known RK4 method in Ch. 1 has the following Butcher tableau:

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

- We have considered this far **so-called explicit schemes**.
- Numerical instability, when the solution includes rapidly varying terms (**stiff problems**).
- Explicit schemes use very small step sizes in order to not diverge from a solution path (computationally demanding).
- In **implicit Runge–Kutta methods**, the Butcher tableau is no longer lower-triangular.
- On every step, **a system of algebraic equations has to be solved** (computationally demanding, but more stable).

- The simplest implicit method is the backward Euler scheme.

Example (Backward Euler)

The implicit backward Euler scheme has the Butcher tableau:

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

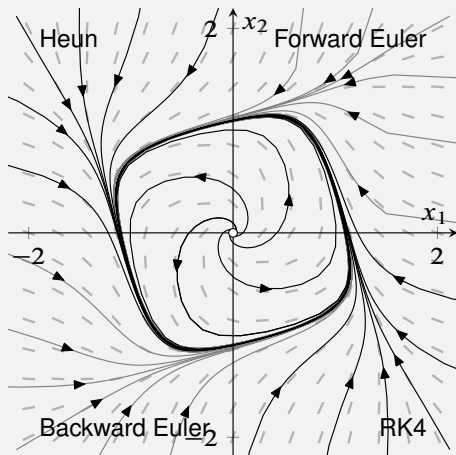
which gives the recursion $\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_{k+1}), t_k + \Delta t) \Delta t$.

- We study the two-dimensional non-linear ordinary differential equation system

$$\dot{x}_1 = x_1 - x_2 - x_1^3,$$

$$\dot{x}_2 = x_1 + x_2 - x_2^3,$$

Example [2/2]



Strong stochastic Runge–Kutta: Basic principles

- A family of **iterative methods** for solving **stochastic differential equations**.
- Based on **Itô–Taylor series** (see the previous lecture), ...
- ... but are **derivative-free**.
- **Plug-and-play** methods that only requires specification of the drift and diffusion function of the SDE (at least ideally).
- We divide the methods into **strong** and **weak** methods (as we did for the Itô–Taylor series approximations).
- **A word of warning:** Stochastic Runge–Kutta methods are not as easy to grasp as the ordinary ones.

- Recall the following multi-dimensional SDE formulation

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}(t), t) dt + \mathbf{L}(\mathbf{x}(t), t) d\boldsymbol{\beta}, \quad \mathbf{x}(t_0) \sim p(\mathbf{x}(t_0)),$$

where the drift is defined by $\mathbf{f} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ and the diffusion coefficients by $\mathbf{L} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d \times \mathbb{R}^m$.

- The driving noise process

$$\boldsymbol{\beta}(t) = (\beta^{(1)}(t), \beta^{(2)}(t), \dots, \beta^{(m)}(t))$$

is an m -dimensional standard Brownian motion.

- In integral form the equation can be expressed as

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}(\mathbf{x}(\tau), \tau) d\tau + \int_{t_0}^t \mathbf{L}(\mathbf{x}(\tau), \tau) d\boldsymbol{\beta}(\tau).$$

- Applying the Itô formula to the terms $\mathbf{f}(\mathbf{x}(t), t)$ and $\mathbf{L}(\mathbf{x}(t), t)$ and collecting the terms gives an Itô–Taylor series expansion of the solution (see the previous lecture):

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(t_0) + \mathbf{f}(\mathbf{x}(t_0), t_0) (t - t_0) + \mathbf{L}(\mathbf{x}(t_0), t_0) (\beta(t) - \beta(t_0)) \\ &\quad + \int_{t_0}^t \int_{t_0}^{\tau} \mathcal{L}_t \mathbf{f}(\mathbf{x}(\tau), \tau) d\tau d\tau \\ &\quad + \sum_i \int_{t_0}^t \int_{t_0}^{\tau} \mathcal{L}_{\beta, i} \mathbf{f}(\mathbf{x}(\tau), \tau) d\beta^{(i)}(\tau) d\tau \\ &\quad + \int_{t_0}^t \int_{t_0}^{\tau} \mathcal{L}_t \mathbf{L}(\mathbf{x}(\tau), \tau) d\tau d\beta(\tau) \\ &\quad + \sum_i \int_{t_0}^t \int_{t_0}^{\tau} \mathcal{L}_{\beta, i} \mathbf{L}(\mathbf{x}(\tau), \tau) d\beta^{(i)}(\tau) d\beta(\tau).\end{aligned}$$

- The first row in the equation is just **the Euler–Maruyama scheme**.
- Similarly as we did in for the ordinary RK methods, we can consider truncated series expansions of various degrees **for each of these terms**.
- The extra terms involving the **iterated and cross-term Itô integrals** complicate the formulation.
- To present a family of actual numerical methods, we consider the following **family of strong order 1.0 methods** due to *Andreas Rößler*...

A class of SRK methods of strong order 1.0 [1/3]

- Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$ into n steps $t_0 < t_1 < t_2 < \dots < t_n = t$ such that $\Delta t = t_{k+1} - t_k$. The integration method is characterized by its *extended Butcher tableau*:

$\mathbf{c}^{(0)}$	$\mathbf{A}^{(0)}$	$\mathbf{B}^{(0)}$	
$\mathbf{c}^{(1)}$	$\mathbf{A}^{(1)}$	$\mathbf{B}^{(1)}$	
	$\boldsymbol{\alpha}^T$	$[\boldsymbol{\gamma}^{(1)}]^T$	$[\boldsymbol{\gamma}^{(2)}]^T$

- On each step k approximate the solution trajectory as follows:

$$\begin{aligned}\hat{\mathbf{x}}(t_{k+1}) = & \hat{\mathbf{x}}(t_k) + \sum_{i=1}^s \alpha_i \mathbf{f}(\tilde{\mathbf{x}}_i^{(0)}, t_k + \mathbf{c}_i^{(0)} \Delta t) \Delta t \\ & + \sum_{i=1}^s \sum_{n=1}^m (\gamma_i^{(1)} \Delta \beta_k^{(n)} + \gamma_i^{(2)} \sqrt{\Delta t}) \mathbf{L}^n(\tilde{\mathbf{x}}_i^{(n)}, t_k + \mathbf{c}_i^{(1)} \Delta t)\end{aligned}$$

A class of SRK methods of strong order 1.0 [3/3]

- With the supporting values

$$\begin{aligned}\tilde{\mathbf{x}}_i^{(0)} &= \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j}^{(0)} \mathbf{f}(\tilde{\mathbf{x}}_j^{(0)}, t_k + c_j^{(0)} \Delta t) \Delta t \\ &\quad + \sum_{j=1}^s \sum_{l=1}^m B_{i,j}^{(0)} \mathbf{L}^l(\tilde{\mathbf{x}}_j^{(l)}, t_k + c_j^{(1)} \Delta t) \Delta \beta_k^{(l)}, \\ \tilde{\mathbf{x}}_i^{(n)} &= \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j}^{(1)} \mathbf{f}(\tilde{\mathbf{x}}_j^{(0)}, t_k + c_j^{(0)} \Delta t) \Delta t \\ &\quad + \sum_{j=1}^s \sum_{l=1}^m B_{i,j}^{(1)} \mathbf{L}^l(\tilde{\mathbf{x}}_j^{(l)}, t_k + c_j^{(1)} \Delta t) \frac{\Delta \beta_k^{(l,n)}}{\sqrt{\Delta t}},\end{aligned}$$

for $i = 1, 2, \dots, s$ and $n = 1, 2, \dots, m$.

The increments [1/2]

- The increments in the algorithm are given by the Itô integrals:

$$\Delta\beta_k^{(i)} = \int_{t_k}^{t_{k+1}} d\beta^{(i)}(\tau) \quad \text{and}$$

$$\Delta\beta_k^{(i,j)} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau_2} d\beta^{(i)}(\tau_1) d\beta^{(j)}(\tau_2),$$

- The increments $\Delta\beta_k^{(i)}$ are independent normally distributed random variables

$$\Delta\beta_k^{(i)} \sim N(0, \Delta t).$$

The increments [2/2]

- The iterated stochastic Itô integrals $\Delta\beta_k^{(i,j)}$ are trickier.
- For these methods, **when $i = j$** , the multiple Itô integrals can be rewritten as

$$\Delta\beta_k^{(i,i)} = \frac{1}{2} \left([\Delta\beta_k^{(i)}]^2 - \Delta t \right).$$

- Exact simulation from the integrals $\Delta\beta_k^{(i,j)}$, **when $i \neq j$** , is **not possible**, but can be approximated.

Example: Euler–Maruyama

Example (Euler–Maruyama Butcher tableau)

The Euler–Maruyama method has the extended Butcher tableau:

$$\begin{array}{c|ccc} 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & \\ \hline & 1 & 1 & 0 \end{array}$$

and as we recall from the previous chapter, it is of strong order 0.5.

Example: A strong order 1.0 method

Example (Strong order 1.0 SRK due to Rößler)

- Consider a stochastic Runge–Kutta method with the following extended Butcher tableau:

0									
1	1			0					
0	0	0		0	0				
0									
1	1			1					
1	1	0		-1	0				
	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$

- The (rather lengthy) algorithm is written out in the lecture notes.

- Higher-order methods by considering **more terms** in the Itô–Taylor expansion.
- **Not very practical in general** (heavy and complicated).
- For models with **some special structure** this might still be feasible:
 - One-dimensional models.
 - Additive noise models.
 - Diagonal noise models.
 - Models with commutative noise.

Example: Duffing van der Pol oscillator [1/4]

- Consider a simplified version of a Duffing van der Pol oscillator

$$\ddot{x} + \dot{x} - (\alpha - x^2)x = x w(t), \quad \alpha \geq 0,$$

driven by multiplicative white noise $w(t)$ with spectral density q .

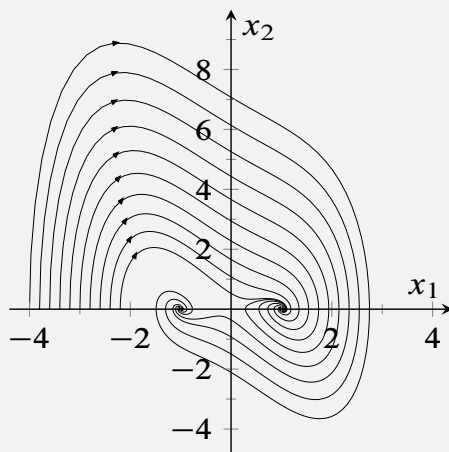
- The corresponding two-dimensional, $\mathbf{x}(t) = (x, \dot{x})$, Itô stochastic differential equation is

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1(\alpha - x_1^2) - x_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ x_1 \end{pmatrix} d\beta,$$

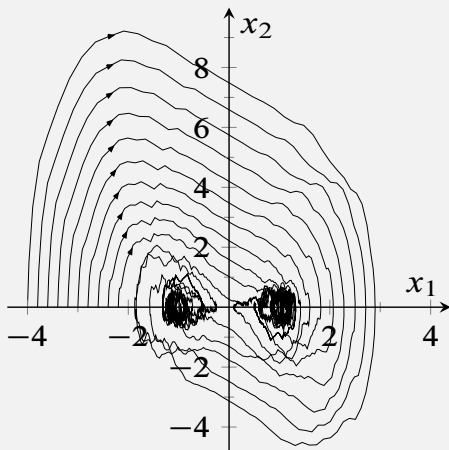
where $\beta(t)$ is a one-dimensional Brownian motion.

- Consider different initializations with $q = 0$ and $q = 0.5^2$. Use the same realizations of noise for each initialization. Use $\alpha = 1$ and $\Delta t = 2^{-5}$.

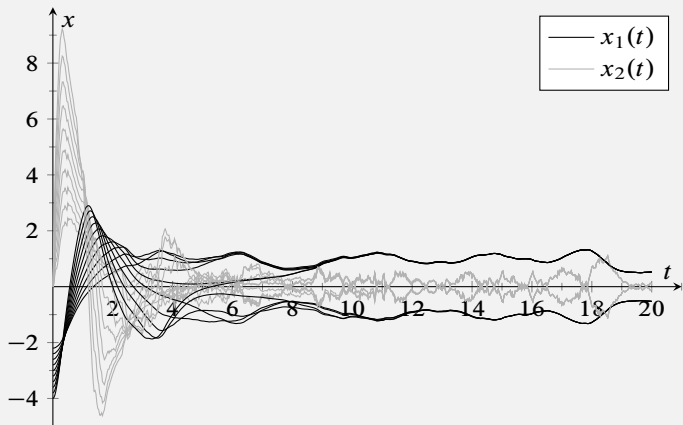
Example: Duffing van der Pol oscillator [2/4]



Example: Duffing van der Pol oscillator [3/4]



Example: Duffing van der Pol oscillator [4/4]



Weak stochastic Runge–Kutta methods

- It is possible to form weak approximations to SDEs, where the **interest is not in the solution trajectories**, but the **distribution** of them.
- We can replace weak Itô–Taylor approximations by Runge–Kutta style approximations which avoid the use of derivatives of the drift and diffusion coefficients.
- The reasoning behind the methods is very much the same as for strong SRK methods.
- Here we consider a rather general **class of weak order 2.0 methods** by Rößler:

A class of SRK methods of weak order 2.0 [1/3]

- Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$ into n steps $t_0 < t_1 < t_2 < \dots < t_n = t$ such that $\Delta t = t_{k+1} - t_k$. The integration method is characterized by the following **extended Butcher tableau**:

$\mathbf{c}^{(0)}$	$\mathbf{A}^{(0)}$	$\mathbf{B}^{(0)}$	
$\mathbf{c}^{(1)}$	$\mathbf{A}^{(1)}$	$\mathbf{B}^{(1)}$	
$\mathbf{c}^{(2)}$	$\mathbf{A}^{(2)}$	$\mathbf{B}^{(2)}$	
	α^T	$[\gamma^{(1)}]^T$	$[\gamma^{(2)}]^T$
		$[\gamma^{(3)}]^T$	$[\gamma^{(4)}]^T$

A class of SRK methods of weak order 2.0 [2/3]

- On each step k approximate the solution by the following:

$$\begin{aligned}\hat{\mathbf{x}}(t_{k+1}) &= \hat{\mathbf{x}}(t_k) + \sum_{i=1}^s \alpha_i \mathbf{f}(\tilde{\mathbf{x}}_i^{(0)}, t_k + \mathbf{c}_i^{(0)} \Delta t) \Delta t \\ &+ \sum_{i=1}^s \sum_{n=1}^m \gamma_i^{(1)} \mathbf{L}^n(\tilde{\mathbf{x}}_i^{(n)}, t_k + \mathbf{c}_i^{(1)} \Delta t) \Delta \hat{\beta}_k^{(n)} \\ &+ \sum_{i=1}^s \sum_{n=1}^m \gamma_i^{(2)} \mathbf{L}^n(\tilde{\mathbf{x}}_i^{(n)}, t_k + \mathbf{c}_i^{(1)} \Delta t) \frac{\Delta \hat{\beta}_k^{(n,n)}}{\sqrt{\Delta t}} \\ &+ \sum_{i=1}^s \sum_{n=1}^m \gamma_i^{(3)} \mathbf{L}^n(\bar{\mathbf{x}}_i^{(n)}, t_k + \mathbf{c}_i^{(2)} \Delta t) \Delta \hat{\beta}_k^{(n)} \\ &+ \sum_{i=1}^s \sum_{n=1}^m \gamma_i^{(4)} \mathbf{L}^n(\bar{\mathbf{x}}_i^{(n)}, t_k + \mathbf{c}_i^{(2)} \Delta t) \sqrt{\Delta t},\end{aligned}$$

A class of SRK methods of weak order 2.0 [3/3]

- With supporting values

$$\begin{aligned}\tilde{\mathbf{x}}_i^{(0)} &= \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j}^{(0)} \mathbf{f}(\tilde{\mathbf{x}}_j^{(0)}, t_k + c_j^{(0)} \Delta t) \Delta t \\ &\quad + \sum_{j=1}^s \sum_{l=1}^m B_{i,j}^{(0)} \mathbf{L}^l(\tilde{\mathbf{x}}_j^{(l)}, t_k + c_j^{(1)} \Delta t) \Delta \hat{\beta}_k^{(l)}, \\ \tilde{\mathbf{x}}_i^{(n)} &= \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j}^{(1)} \mathbf{f}(\tilde{\mathbf{x}}_j^{(0)}, t_k + c_j^{(0)} \Delta t) \Delta t \\ &\quad + \sum_{j=1}^s \sum_{l=1}^m B_{i,j}^{(1)} \mathbf{L}^l(\tilde{\mathbf{x}}_j^{(l)}, t_k + c_j^{(1)} \Delta t) \Delta \hat{\beta}_k^{(l,n)}, \\ \bar{\mathbf{x}}_i^{(n)} &= \hat{\mathbf{x}}(t_k) + \sum_{j=1}^s A_{i,j}^{(2)} \mathbf{f}(\tilde{\mathbf{x}}_j^{(0)}, t_k + c_j^{(0)} \Delta t) \Delta t \\ &\quad + \sum_{j=1}^s \sum_{\substack{l=1 \\ l \neq n}}^m B_{i,j}^{(2)} \mathbf{L}^l(\tilde{\mathbf{x}}_j^{(l)}, t_k + c_j^{(1)} \Delta t) \frac{\Delta \hat{\beta}_k^{(l,n)}}{\sqrt{\Delta t}},\end{aligned}$$

The increments [1/2]

- Again, the increments are given by the **double Itô integrals**.
- In the weak schemes we can use the following approximations:

$$\Delta \hat{\beta}_k^{(i,j)} = \begin{cases} \frac{1}{2} (\Delta \hat{\beta}_k^{(i)} \Delta \hat{\beta}_k^{(j)} - \sqrt{\Delta t} \hat{\zeta}_k^{(i)}), & \text{if } i < j, \\ \frac{1}{2} (\Delta \hat{\beta}_k^{(i)} \Delta \hat{\beta}_k^{(j)} + \sqrt{\Delta t} \hat{\zeta}_k^{(j)}), & \text{if } i > j, \\ \frac{1}{2} ([\Delta \hat{\beta}_k^{(i)}]^2 - \Delta t), & \text{if } i = j. \end{cases}$$

- Here only **$2m - 1$ independent random variables** are needed.
- **No problems** with the cross-term integrals any more.

The increments [2/2]

- For example, we can choose $\Delta\hat{\beta}_k^{(i)}$ such that they are independent **three-point distributed** random variables:

$$\mathbb{P}(\Delta\hat{\beta}_k^{(i)} = \pm\sqrt{3\Delta t}) = \frac{1}{6} \quad \text{and} \quad \mathbb{P}(\Delta\hat{\beta}_k^{(i)} = 0) = \frac{2}{3},$$

- The supporting variables $\hat{\zeta}_k^{(i)}$ such that they are independent **two-point distributed** random variables.

$$\mathbb{P}(\hat{\zeta}_k^{(i)} = \pm\sqrt{\Delta t}) = \frac{1}{2}.$$

Example: A weak order 2.0 method

Example (Weak order 2.0 SRK due to Rößler)

- Consider a stochastic Runge–Kutta method with the following extended Butcher tableau:

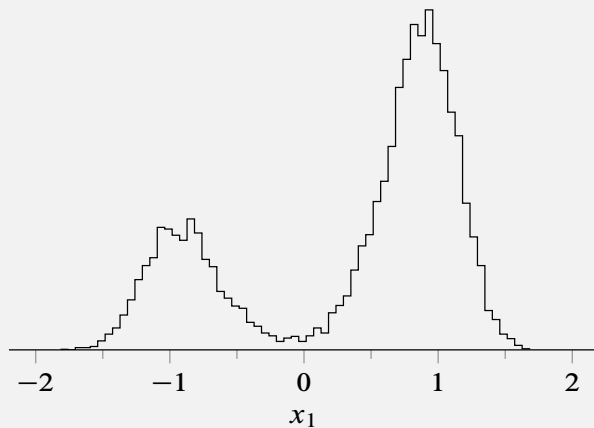
0									
1	1			1					
0	0	0		0	0				
0									
1	1			1					
1	1	0		-1	0				
0									
1	1			1					
1	1	0		-1	0				
	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
				$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$

- The (rather lengthy) algorithm is written out in the lecture notes.

Example: Weak SRK for Duffing van der Pol [1/2]

- We are interested in characterizing the solution at $t = 20$ for the initial condition of $\mathbf{x}(0) = (-3, 0)$.
- We use the stochastic Runge–Kutta method of weak order 2.0.
- Discretization interval: $\Delta t = 2^{-4}$.
- We show the results as a histogram of $x_1(20)$ with 10,000 samples.
- With a Δt this large, the Euler–Maruyama method does not provide plausible results.

Example: Weak SRK for Duffing van der Pol [2/2]



Summary

- Stochastic Runge–Kutta methods are **derivative-free methods** for solving SDEs.
- They **cannot be derived as simple extensions** to ordinary Runge–Kutta methods.
- You **cannot get rid of the iterated Itô integral**.
- The **complexity** of the methods grows with the approximation order.
- Higher order schemes can be practical for models with some **special structure** (scalar, additive, commutative, *etc.*).
- The choice between a **weak** and **strong** scheme depends on your application.