Mathematics for Economists
ECON-C1000
Spring 2020
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Problem Set 5: Due May 20, 2020

1. Consider the UMP with Cobb-Dougls utility function $u(x, y) = x^{\alpha}y^{1-\alpha}$ for $0 < \alpha < 1$ on $\mathbb{R}^2_+$. The budget constraint is $p_x x + p_y y \leq w$.

   (a) Argue using Weierstrass' theorem that the problem has a solution, that any point satisfying K-T -conditions is optimal, and that all optimal solutions are interior and budget constraint binds.

   (b) Set $MRS_{x,y} = \frac{p_x}{p_y}$ and solve for optimal y as a function of x.

   (c) Plug into budget constraint and find optimal solutions for x, y. Check from K-T conditions that the multiplier on the budget constraint is non-negative.

   (d) Plug the solutions to objective function to get the indirect utility function.

   (e) From the indirect utility function, derive the optimal demands using Roy's identity.

2. A consumer lives for two periods $t \in \{1, 2\}$ and gets utility $u_t(c_t)$ from consuming $c_t \geq 0$ in period t so that her total utility is $u(c_1, c_2) = u_1(c_1) + u_2(c_2)$. In the first period, she has income $w_1 \geq 0$ and in the second, she has income $w_2 \geq 0$. Assume that $u_t'(c_t) > 0$ and $u_t''(c_t) < 0$ for $t \in \{1, 2\}$.

   (a) What is the feasible set of consumptions (c_1, c_2) for a consumer that can save income (at zero interest) but cannot borrow against future income?

   (b) What is the budget set if the consumer can also borrow (at zero interest) against future income so that any borrowings in $t = 1$ must be paid back from $t = 2$ income?

   (c) For the case where the two utility functions are identical so that $u_1(c_t) = u_2(c_t) = u(c_t)$, set up the constrained optimization problem and solve it when borrowing and saving is possible.

   (d) How does your answer change if the consumer is impatient so that $u_2(c) = \delta u_1(c)$ for some $0 < \delta < 1$?

   (e) When can you have different marginal utilities of consumption across the two periods at optimum in the case where the utility functions are identical and you can only save?

3. Continue with he same setting and now assume that $u(c_1, c_2) = \ln c_1 + \delta \ln c_2$ with $0 < \delta < 1$ for $c_1, c_2 > 0$ and $u(c_1, c_2) = -\infty$ if $c_t = 0$ for some t. Assume that saving and borrowing is possible at interest rate r so that by saving s in $t = 1$, you get back $(1+r)s$ in $t = 2$. If you borrow b in $t = 1$, you must pay back $(1+r)b$ in $t = 2$. (Note that $w_1 - c_1$ is your savings (if it is positive) and borrowing (if it is negative)).

   (a) Set up the consumer's problem and its Lagrangean. In particular, derive a single budget constraint for the consumptions. Derive the first-order K-T conditions.

   (b) Argue that the K-T conditions are sufficient so that any solution to the K-T conditions solves the maximization problem.

   (c) Determine which constraints must bind and solve the problem for optimal consumptions (c_1^*, c_2^*).

   (d) When is $c_1^* < c_2^*$? Does this depend on the difference $w_1 - w_2$?

4. Continue with the utility function from the previous problem, but assume now (realistically) that you can save at interest rate $\underline{r}$ and borrow at $\overline{r}$ with $\overline{r} > \underline{r}$.

   (a) Draw the budget. set for the consumer in the (c_1, c_2) coordinates.

   (b) Find bounds for the consumer's MRS_{c_1,c_2} at optimum based on the picture (recall that utility is unboundedly negative as $c_t \to 0$).

   (c) What is the optimality condition for the consumers that save? What is the optimality condition for those who borrow?

   (d) What can be said about the MRS of those consumers that neither save nor borrow?

   (e) Solve the optimization problem.

5. Return to the case where saving and borrowing takes place at the same rate. Suppose that there are two types of consumers that you may have encountered before in your studies. There are n students with $w_1^s = 0, w_2^s = 70$ and m trust fund kids with $w_1^t = 100, w_2^t = 0$.

   (a) Plug the incomes of the two types of consumers into your solution to problem 2 and compute total savings by trust fund kids and total borrowing of students at interest rate r.

   (b) In equilibrium, what I borrow, someone else must save. Therefore the total savings of trust fund kids must equal total borrowing by the students. Treat the interest rate as the endogenous variable to balance the market. I.e. find r^* such that
$$nc_1^s(r^*) = m(w_1^t - c_1^t(r^*)).$$

   (c) How does the equilibrium interest rate depend on n, m, w_1^t? What are the effects of w_1^t on the welfare (optimal utility at equilibrium r^*) of the two types of consumers?