Mathematics for Economists: Lecture 11

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This lecture covers

1. What are dynamical systems?
2. Difference equations and motivating examples
3. Linear difference equations with constant coefficients

Dynamical systems

- Dynamical systems describe the evolution of variables over time
- For the state so the system today, x_t we can determine the state tomorrow x_{t+1}

$$x_{t+1} = f(x_t).$$

- The solution is a sequence $\{x_t\}$ satisfying this equation for all $t + 1, t$.
- Hence the variable to be determined is the entire path of x.

Dynamical systems

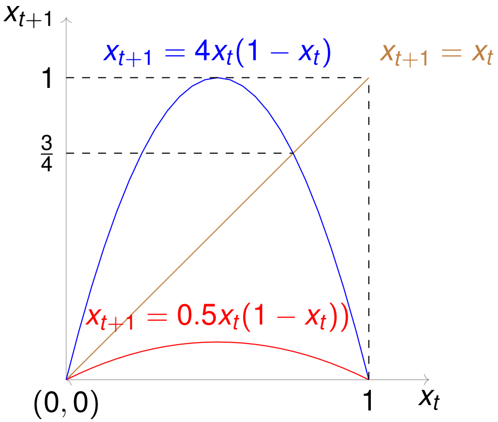
- What are the interesting questions with dynamical systems
- Is there a steady state, i.e. a value x^* such that $x^* = f(x^*)$?
- Do the solutions converge to this steady state?
- Are the solutions monotone?
- Can we have cycles?
- Because of time constraints, we cannot go very deep into this

Dynamical systems: logistic equation

- How to picture a dynamical system? We start with the case where $x_t \in \mathbb{R}$.
- Start with a concrete example:

$$x_{t+1} = rx_t(1 - x_t).$$

- This is a nice differentiable function whose values remain in $(0, 1)$ for all t as long as $r < 4$.
- To analyze a difference equation on the real line, the first step is to look at the graph of the system equation.



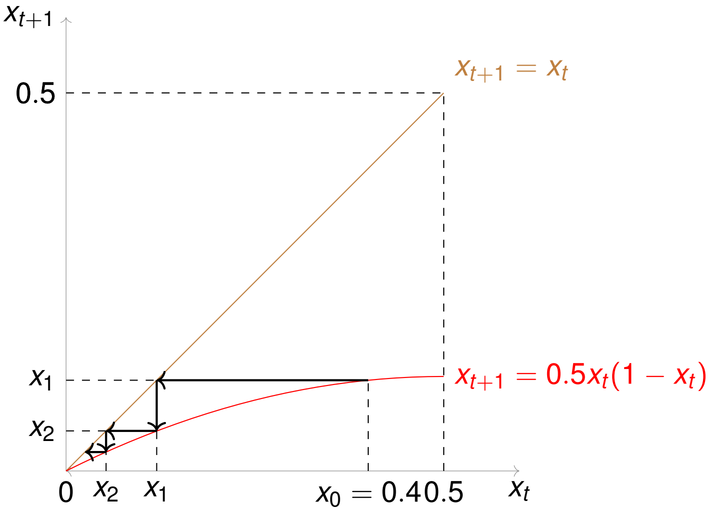
x_{t+1}
$x_{t+1} = 4x_t(1 - x_t)$
$x_{t+1} = x_t$
1
$\frac{3}{4}$
$x_{t+1} = 0.5x_t(1 - x_t))$
(0, 0)
1
x_t

Dynamical systems: logistic equation

- What is the significance of the intersections of $x_{t+1} = rx_t(1 - x_t)$ and the 45 -degree line $x_{t+1} = x_t$.
- The system stops at any such point, because if $x_{t+1} = x_t$, then also $x_{t+k} = x_t$ by repeated substitution into the system equation.
- These are called the steady states of the dynamical system.
- Notice that the system has a single steady state at $x = 0$ if $r < 1$ (can you show this?).
- For $4 > r > 1$, the system has another steady state at $x = \frac{r-1}{r}$. What happens to the values of x_t as t grows?

Dynamical systems: logistic equation

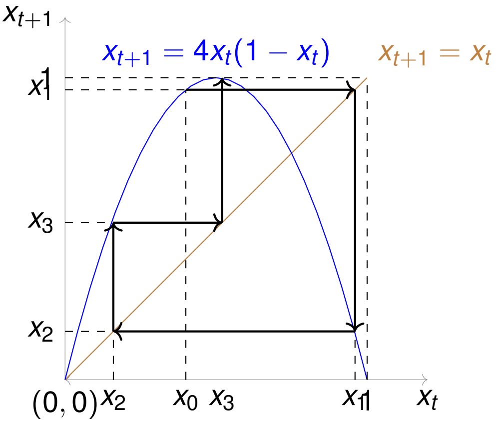
- Here is a nice graphical way of seeing what happens to the sequence.
- Lets graph the function in a coordinate system where x_{t+1} is on the vertical and x_t is on the horizontal axis.
- Draw the graph of $x_{t+1} = f(x_t)$ and pick a starting point $x_0 = 0.4$ for example on the horizontal axis.
- You can read $x_1 = f(x_0)$ on the graph.
- You need to picture x_1 on the horizontal axis to see where x_2 is located. But you can do this by reflection through the 45 -degree line. The you just continue the procedure.
- Lets look first at the case $x_t + 1 = \frac{1}{2}x : x(1 - x_t)$, i.e. lets take the red curve in the previous picture.



x_{t+1}
$x_{t+1} = x_t$
0.5
x_1
x_2
$x_{t+1} = 0.5x_t(1 - x_t)$
0
x_2
x_1
$x_0 = 0.4$
0.5
x_t

Dynamical systems: logistic equation

- As you can see, for any starting x_0, the system x_t converges quite quickly to 0.
- If we take the blue graph from the first picture, things look quite different.
- Let's follow the system again for a few rounds starting at $x_0 = 0.4$.


x_{t+1}
$x_{t+1} = 4x_t(1 - x_t)$
$x_{t+1} = x_t$
1
x_1
x_3
x_2
(0, 0)
x_2
x_0
x_3
x_1
1
x_t

Motivating examples: the Solow growth model

- In this simplest version of the model, labor is kept fixed at L over time and capital K_t changes over time as a result of savings.
- The aggregate production function is $y_t = F(K_t, L_t)$ and it is usually assumed to be an increasing concave, linearly homogenous (constant returns to scale) so that

$$F(K_t, L_t) = L_t F(\frac{K_t}{L_t}, 1) := L_t f(k_t) = Lf(k_t),$$

where $k_t := \frac{K_t}{L_t}$.
- It is often assumed that $\lim_{k \to 0} = \infty$ and $\lim_{k \to \infty} = 0$.
- Let's assume here that $y_t = k_t^{\alpha}$ for some $0 < \alpha < 1$.

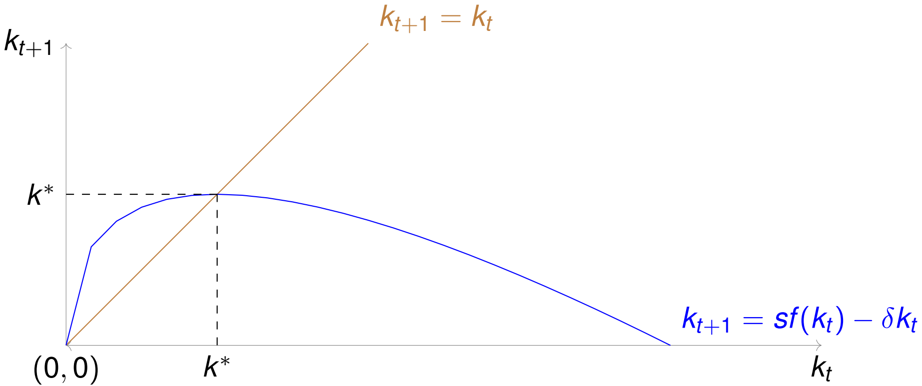
Motivating examples: the Solow growth model

- The output y_t is divided between savings and consumption.
- The assumption is that a constant fraction sy_t is saved.
- As you can recall from Principles of Economics II, savings equals investment and investment goes into next period's capital.
- Capital depreciates at rate δ per period.
- Taking all this together, we get

$$k_{t+1} = sk_t^{\alpha} - \delta k_t.$$

Motivating examples: the Solow growth model

- Let's draw the graph of $sk_t^\alpha - \delta k_t$ in the (k_t, k_{t+1}) plane with the 45-degree line.
- Of special interest is the point k^* such that $s(k^*)^{alpha} - \delta k^* = k^*$.
- If you start the system with $k_0 = k^*$, the system stays there forever since $k_1 = sk_0^\alpha - \delta k_0 = k_0$ and therefore also $k_t + n = k_0$ for all n.
- We call k^* the steady state or a rest point of the system.


$k_{t+1} = k_t$
k_{t+1}
k^*
$k_{t+1} = sf(k_t) - \delta k_t$
$(0,0)$
k^*
k_t

Motivating examples: the Solow growth model

- We can picture the movement of the system by positing an initial point k_0 on the horizontal axis.
- If $k_0 < k^*$, then $k_1 = f(k_0) > k_0$.
- You can locate the k_1 on the horizontal axis by reflecting on the 45-degree line.
- Repeating this process, you can show that from any initial point, k_t converges to k^* as $t \to \infty$.
- We say that k^* is a globally stable steady state.
- You may want to note that

$$k^* = (\frac{s}{1+\delta})^{\frac{1}{1-\alpha}}.$$

Motivating examples: Fibonacci sequence

- Consider the sequence of numbers formed by the rule

$$x_{t+2} = x_{t+1} + x_t,$$

  and set $x_0 = 0, x_1 = 1$.
- What is the sequence of numbers generated by this rule?
- This sequence is called the Fibonacci sequence and it is one of the most famous sequences in all of mathematics
- Notice that we have now dependence on two past values, but we can deal with this by letting $y_{t+1} = x_t$ and considering the system

$$(x_{t+1}, y_{t+1}) = A(x_t, y_t)$$

  for a suitably chosen A.
- We'll see how to do this tomorrow.

Motivating examples: SIR model of an epidemic

- Let the population be divided into three classes. Susceptible s_t, infected i_t, and recovered r_t.

$$\begin{aligned} s_{t+1} &= s_t - \beta i_t s_t, \\ i_{t+1} &= \beta i_t s_t - \alpha i_t, \\ r_{t+1} &= \alpha i_t. \end{aligned}$$

- Here $\beta > 0$ is the infection rate in meetings and $\alpha > 0$ is the recovery rate.
- The much talked about R_0 is simply $\frac{\beta}{\alpha}$.

Motivating examples: SIR model of an epidemic

- I guess I do not have to convince you of the importance of this model now.
- This is a non-linear model that does not have an easy closed form solution
- You can compute the steady state fractions of susceptible and recovered when the epidemic comes to an end.
- Can you find this as a function of R_0?
- You can see on Youtube a number of nice expositions on how to simulate an epidemic. For example a good one is on 3blue1brown at https://www.youtube.com/watch?v=gxAaO2rsdIs

Markov process

- A population consists of three income classes $i \in \{1, 2, 3\}$.
- If you are in class i, your children are in income class j with probability p_{ji}. Let P be the matrix with a typical element p_{ij}.
- Let $x_0 = \mathbf{e_i}$ if you are in class. Then the probability that you child is in class j is given by the column vector

$$x_1 = Px_0 = P\mathbf{e_i}.$$

- But then the probability that your grandchild is in class j is given by the column vector

$$x_2 = Px_1 = P^2 x_0,$$

and in general,

$$x_{t+1} = Px_t.$$

Linear difference equations in $\mathbb{R}$

- The simplest form of difference equations are linear difference equations with constant coefficients. These can be written as:

$$x_{t+1} = Ax_t + b_t,$$

  where b_t is a given sequence.
- If $b_t = 0$ for all t, we have a homogenous equation. We start with the simplest homogenous equations where $x_t \in \mathbb{R}$ and $A = a \in \mathbb{R}$.
- Solving the homogenous equation is very easy. If $x_{t+1} = ax_t$ for all t, then $x_{t+k} = a^k x_t$.
- Hence any sequence of the form $x_t = ca^t$ solves the difference equation.
- If we are given the initial value x_0, the solution is $x_t = x_0 a^t$.

Linear difference equations in $\mathbb{R}$

- In other words, the initial value pins down the coefficient c of the general solution.
- Consider next an inhomogenous equation,

$$x_{t+1} = ax_t + b,$$

  where $b_t = b$ for all t.
- Clearly the constant solution $x_t = \frac{b}{1-a}$ for all t solves the equation.
- I claim that also $x_t = ca^t + \frac{b}{1-a}$ solves the equation. But this follows immediately from the fact that $ca^{t+1} = aca^t$.
- This principle holds more generally. If you have a particular solution x_t^P to the inhomogenous equation and the general solution of the homogenous equation x_t^H, then the general solution to the problem is $x_t^P + x_t^H$.
- This is called the principle of superposition and it arises from the linearity of the equations in x_{t+1}, x_t. It is valid also for the case with $x_t \in \mathbb{R}^t$.